

BRACKETS*

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Dedicated to the memory of Jean-Louis Loday

Abstract

We review origins and main properties of the most important bracket operations appearing canonically in differential geometry and mathematical physics in the classical, as well as in the supergeometric setting. The review is supplemented by some new concepts and examples.

1 Introduction

In algebra, ‘brackets’ are usually understood as non-associative operations on vector spaces or modules, with Lie brackets as the main example. The aim of this notes is to present a survey of bracket operations playing an important role in geometry and physics applications. What we will consider are mainly canonical Lie and, more generally, Loday brackets in the standard, as well as superalgebraic setting. Among them are Poisson and Jacobi (more generally, Kirillov) brackets, Schouten-Nijenhuis, Nijenhuis-Richardson, and Frölicher-Nijenhuis brackets, Lie algebroid brackets, Courant and Dorfman brackets, n -ary brackets of Filippov and Nambu, etc.

A proper understanding of the roots and properties of all these brackets requires a basic knowledge of superalgebra and graded differential geometry whose rudiments will be also outlined in these notes. We will add also a few new concepts and examples. Of course, the subject is so extensive that we are only able to sketch selected problems and cite only a small part of the existing literature. We hope, however, that this review could be of some interest for those who encounter the brackets in their work with problems of contemporary mathematics and physics.

2 Lie and graded Lie algebras

2.1 Algebras

By an *algebra* on a vector space \mathcal{A} over a field \mathfrak{k} we will understand a bilinear operation on \mathcal{A} ,

$$\mathcal{A} \times \mathcal{A} \ni (x, y) \mapsto x \circ y \in \mathcal{A}. \quad (1)$$

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In most cases we can consider as well algebras over commutative rings. The algebra (\mathcal{A}, \circ) we call *commutative* if the operation is commutative, $x \circ y = y \circ x$, and *anti-commutative* or *skew-symmetric* if $x \circ y = -y \circ x$. We call (\mathcal{A}, \circ) *unital* if it has a *unit*, i.e. an element $\mathbf{1} \in \mathcal{A}$ for which $\mathbf{1} \circ x = x \circ \mathbf{1} = x$. Note that the unit is unique if it exists. A commutative associative operation we will usually denote " \cdot ", or even write it simply as the juxtaposition, e.g. xy .

If $\{x_i\}_{i \in I}$ is a basis of \mathcal{A} , the algebra structure is uniquely determined by the *structure constants* c_{ij}^k , where (the summation convention is used)

$$x_i \circ x_j = c_{ij}^k x_k. \quad (2)$$

Example 2.1. Let us observe that any vector space V gives rise to a canonical nontrivial algebra structure. Namely, the space $\mathcal{A} = \mathfrak{gl}(V) = \text{End}_{\mathbb{F}}(V)$ of all linear endomorphisms $x : V \rightarrow V$ is an algebra with the operation being just the composition of maps. This algebra is *associative*, i.e. the map

$$m : \mathcal{A} \rightarrow \mathfrak{gl}(\mathcal{A}), \quad m_x(y) = x \circ y, \quad (3)$$

is an algebra homomorphism,

$$m_{x \circ y} = m_x \circ m_y. \quad (4)$$

Here, of course, the first " \circ " is the operation in \mathcal{A} and the second in $\mathfrak{gl}(\mathcal{A})$. In other words, the product in \mathcal{A} satisfies the identity

$$(x \circ y) \circ z = x \circ (y \circ z). \quad (5)$$

We call m the *(left) regular representation* of (\mathcal{A}, \circ) .

Example 2.2. If M is a topological space, then the set $C(M)$ of all real continuous functions on M is canonically a commutative associative algebra over \mathbb{R} with the point-wise multiplication. Similarly, if M is a smooth manifold, then the set $C^\infty(M)$ of all real smooth functions on M is also a commutative associative algebra.

Having one binary operation, we can easily produce other operations. For instance, we can consider the *commutator* $[x, y] = x \circ y - y \circ x$, or the *anti-commutator* (*symmetrizer*) $x \vee y = x \circ y + y \circ x$. For linear operators, this produces canonical, this time a skew-symmetric (resp., symmetric), operation in $\mathcal{A} = \mathfrak{gl}(V)$ which is in general no longer associative; for the symmetric product we have only a weak version of the associativity:

$$(x \vee y) \vee (x \vee x) = x \vee (y \vee (x \vee x)). \quad (6)$$

These structures are prototypes of what we call a *Lie algebra* or, respectively, a *Jordan algebra*. For instance, we can easily check the following analog of (4):

$$\text{ad}_{[x, y]} = [\text{ad}_x, \text{ad}_y], \quad (7)$$

where $\text{ad} : \mathcal{A} \rightarrow \mathfrak{gl}(\mathcal{A})$ is the corresponding regular representation for the commutator, $\text{ad}_x(y) = [x, y]$. In other words, ad is a homomorphism of the brackets, i.e.

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]]. \quad (8)$$

Identity (8) we call the *Jacobi identity*.

Remark 2.3. Let us remark that sometimes by the Jacobi identity one understands the identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \quad (9)$$

For a skew-symmetric operation (bracket) both versions are equivalent, but for brackets which are not skew-symmetric this is no longer true. The advantage of (8) is its clear algebraic meaning: ad is a representation; therefore the Jacobi identity will be for us always (8).

The Jacobi identity means, equivalently, that operators ad_x are derivations of the bracket. Recall that a *derivation* of an algebra (\mathcal{A}, \circ) is a map $D \in \mathfrak{gl}(\mathcal{A})$ such that, for all $x, y \in \mathcal{A}$,

$$D(x \circ y) = D(x) \circ y + x \circ D(y), \quad (10)$$

i.e. the *Leibniz rule* is satisfied. A trivial but very useful observation giving a method of constructing derivations is that, if \mathcal{A} is freely generated by (x_i) and $y_i \in \mathcal{A}$, then there is a unique derivation D of \mathcal{A} such that $D(x_i) = y_i$.

A bracket $[\cdot, \cdot]$ satisfying the Jacobi identity is called a *Leibniz bracket* or *Loday bracket* and the corresponding algebra a *Leibniz (Loday) algebra*. This terminology goes back to the work of J.-L. Loday who discovered that one can skip the skew-symmetry assumption in the definition of a Lie algebra, still having a possibility to define appropriate (co)homology (see [65, 66] and [64, Chapter 10.6]). Loday himself called these structures *Leibniz algebras*.

Of course, if the Loday bracket is additionally skew-symmetric, we speak about a *Lie bracket* and a *Lie algebra*. The space $\mathcal{A} = \mathfrak{gl}(V)$ of linear operators with the commutator bracket is therefore a canonical example of a Lie algebra.

Similarly, a vector space equipped with a symmetric operation \vee satisfying (6) we call a *Jordan algebra*. The space $\mathcal{A} = \mathfrak{gl}(V)$ of linear operators with the anti-commutator bracket is therefore a canonical example of a Jordan algebra. Operators symmetric with respect to a certain anti-involution $x \mapsto x^\dagger$ (think of Hermitian operators) form a Jordan subalgebra in $\mathfrak{gl}(V)$.

If V has additionally an algebra structure with respect to a product " \circ ", we can distinguish canonically a Lie subalgebra in $\mathcal{A} = \mathfrak{gl}(V)$. This is one of the major ways of obtaining Lie algebra structures.

Proposition 2.4. *For any algebra (\mathcal{A}, \circ) , the space $\text{Der}(\mathcal{A}, \circ)$ of derivations of \mathcal{A} is a Lie subalgebra in $\mathfrak{gl}(\mathcal{A})$ with respect to the commutator bracket.*

If the product " \circ " is fixed, instead of $\text{Der}(V, \circ)$ we will write simply $\text{Der}(V)$.

Example 2.5. Let M be a manifold and let $\mathcal{A} = C^\infty(M)$ be the commutative associative algebra of smooth functions on M . Then, the Lie algebra $\text{Der}(\mathcal{A})$ is canonically identified with the Lie algebra $\mathcal{X}^1(M)$ of smooth vector fields on M with the Lie bracket of vector fields.

Problem. Show that there are no non-zero derivations of the algebra $C(\mathbb{R})$ of all continuous functions on \mathbb{R} , i.e. the differential calculus for $\mathcal{A} = C(\mathbb{R})$ is trivial.

2.2 Modules

Having a (commutative) associative, (resp., Leibniz, Lie, Jordan, etc.) algebra (A, \circ) over \mathfrak{k} , we define its *module* as a vector space V over \mathfrak{k} equipped with two operations, $A \times V \ni (a, v) \mapsto a \circ v \in V$ and $V \times A \ni (v, a) \mapsto v \circ a \in V$, such that $A \oplus V$ becomes also a (commutative) associative, (resp., Leibniz, Lie, Jordan, etc.) algebra with obviously defined operation, denoted with some abuse of notation also \circ , which is trivial on V , $v_1 \circ v_2 = 0$.

Example 2.6. If (A, \cdot) is a commutative algebra, then its module V is defined by a multiplication $A \times V \ni (a, v) \mapsto av \in V$ such that $(a_1 a_2)v = a_1(a_2 v)$. The other multiplication $V \times A \ni (v, a) \mapsto va \in V$ is uniquely determined by the symmetry, $va = av$.

Example 2.7. If $\tau : E \rightarrow M$ is a vector bundle, then the space $\mathcal{E} = \text{Sec}(E)$ of all sections of E is canonically an $\mathcal{A} = C^\infty(M)$ -module with the obvious multiplication (\mathcal{A} is commutative, so the left and the right multiplications are equal) of section by functions on M .

Example 2.8. In particular, the space $\mathcal{X}^1(M)$ of vector fields on M is a $C^\infty(M)$ -module and the canonical Lie bracket of vector fields is related to this module structure by the following ‘Leibniz rule’:

$$[X, fY] = f[X, Y] + X(f)Y. \quad (11)$$

Actually, the action of vector fields on functions makes $C^\infty(M)$ into a $\mathcal{X}^1(M)$ -module, so that $X(f)$ can be viewed as a bracket, $[X, f]$, which makes the space

$$\mathcal{X}^1(M) \oplus C^\infty(M) = \text{Sec}(TM \times \mathbb{R}) \quad (12)$$

of linear first-order differential operators on M into a Lie algebra.

Note that the isomorphism class of the Lie algebra $\mathcal{X}^1(M)$ completely determines M up to a diffeomorphism, exactly like does it the associative algebra $C^\infty(M)$ [22, 26, 89]. Similar results are valid also for the Lie algebras of first-order differential operators [36], Kirillov’s local Lie algebras [28], and for supermanifolds [30].

2.3 Graded algebras

Let K be a commutative associative ring with identity, $U(K)$ be the group of invertible elements of K , and let G be a commutative semigroup. A map $\varepsilon : G \times G \rightarrow U(K)$ is called a *factor* on G if

$$\varepsilon(g, h)\varepsilon(h, g) = 1, \quad p(g) = \varepsilon(g, g) = \pm 1, \quad \text{and} \quad \varepsilon(f, g+h) = \varepsilon(f, g)\varepsilon(f, h), \quad (13)$$

for all $f, g, h \in G$. Let V be a G -graded K -algebra, $V = \bigoplus_{g \in G} V^g$. Elements x from V^g we call *homogeneous of degree g* (or *of weight g*) and denote $g = w(x)$. The algebra V is called *ε -commutative* if

$$a \circ b = \varepsilon(w(a), w(b))b \circ a \quad (14)$$

for all G -homogeneous elements $a, b \in V$. Homogeneous elements a with $p(w(a)) = -1$ we call *odd*, the other homogeneous elements we call *even*. In what follows, K will be \mathbb{R} and ε will take the form $\varepsilon(g, h) = (-1)^{\langle g|h \rangle}$ for a ‘scalar product’ $\langle \cdot | \cdot \rangle : G \times G \rightarrow \mathbb{Z}$, with $G = \mathbb{Z}_2$ (superalgebra case), $G = \mathbb{Z}^n$, or $G = \mathbb{N}^n$. This means that we use the factor as a *sign rule* which can be applied (separately) to any axiom: we change the sign by $(-1)^{\langle g|h \rangle}$ whenever two consecutive homogeneous elements $x \in V^g$ and $y \in V^h$ are interchanged.

We say that our operation “ \circ ” is *of degree k* (or simply *even* or *odd* for $k = 0, 1 \in G = \mathbb{Z}_2$) if

$$V^g \circ V^h \subset V^{g+h+k}. \quad (15)$$

For operations of degree k it is natural to consider the sign rules in the form

$$a \circ b = (-1)^{\langle w(a)+k|w(b)+k \rangle} b \circ a. \quad (16)$$

If G is a group, we can always work with degree 0, making the corresponding shift, $V[k]$, in the grading, where $V[k]^i = V^{i-k}$. For \mathbb{Z}^n -gradings, we will use by default $\langle i|j \rangle = i_1 j_1 + \dots + i_n j_n$,

but other bi-additive pairings $\mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$ are also acceptable. In this way we get the graded versions (of degree $k \in \mathbb{Z}^n$) of our structures. For instance, the graded symmetry of degree k reads

$$x \circ y = (-1)^{\langle w(x)+k|w(y)+k \rangle} y \circ x, \quad (17)$$

the graded skew-symmetry of degree k reads

$$x \circ y = -(-1)^{\langle w(x)+k|w(y)+k \rangle} y \circ x, \quad (18)$$

and the graded Jacobi identity of degree k

$$[[x, y], z] = [x, [y, z]] - (-1)^{\langle w(x)+k|w(y)+k \rangle} [y, [x, z]]. \quad (19)$$

If $k = 0$, we speak simply about the graded commutativity, the graded skew-symmetry, the graded Jacobi identity, etc. The above identities for the degree k are just the graded commutativity, graded skew-symmetry and the graded Jacobi identity for the shifted grading. Note that no sign appears in the associativity property (5).

Example 2.9. Starting with a vector bundle $\tau : E \rightarrow M$ we can consider the *tensor algebra* of its sections,

$$\text{Ts}(E) = \text{Sec}(\otimes E) = \oplus_{i=0}^{\infty} \text{Sec}(E^{\otimes i}), \quad (20)$$

which is clearly an associative (but noncommutative) graded algebra with respect to the tensor product. It contains the *Grassmann algebra*

$$\mathbb{A}(E) = \oplus_{i=0}^{\infty} \mathbb{A}^i(E) = \oplus_{i=0}^{\infty} \text{Sec}(\wedge^i E) \quad (21)$$

which consists of skew-symmetric tensors and is a graded-commutative associative algebra with respect to the wedge product \wedge , the skew-symmetrization of the tensor product. In particular, for $E = \text{T}M$ and $E = \text{T}^*M$, we obtain the \mathbb{N} -graded commutative associative algebras $\mathcal{X}(M) = \oplus_{i=0}^{\infty} \mathcal{X}^i(M)$ and $\Omega(M) = \oplus_{i=0}^{\infty} \Omega^i(M)$ of multivector fields and differential forms, respectively.

Given a graded algebra $(A = \oplus_{i \in \mathbb{Z}^n} A^i, \circ)$, we define the space $\text{Der}^k(A)$ of *graded derivations* of degree $k \in \mathbb{Z}^n$ as the space of maps $D : A \rightarrow A$ of degree k such that

$$D(x \circ y) = D(x) \circ y + (-1)^{\langle w(x)|k \rangle} x \circ D(y) \quad (22)$$

is satisfied. Then, the graded space of (graded) derivations $\text{Der}(A) = \oplus_{k \in \mathbb{Z}^n} \text{Der}^k(A)$ is a graded Lie algebra with respect to the graded commutator,

$$[D_1, D_2] = D_1 D_2 - (-1)^{\langle k_1|k_2 \rangle} D_2 D_1, \quad (23)$$

where $D_i \in \text{Der}^{k_i}(A)$, $i = 1, 2$.

Problem. What are graded derivations of degree 0 of the Grassmann algebra $\Omega(M)$ of differential forms?

Note that for a skew-symmetric bracket operation $[\cdot, \cdot] : V \wedge V \rightarrow V$ on a, say, finite-dimensional vector space V , its dual is a certain map $d : V^* \rightarrow V^* \wedge V^*$. As the Grassmann algebra $\mathbb{A}(V^*)$ is freely generated by any basis of V^* , this maps gives rise to a uniquely defined graded derivation $d : \mathbb{A}(V^*) \rightarrow \mathbb{A}(V^*)$ of degree 1. Indeed, we can inductively define

$$d(v_0 \wedge \cdots \wedge v_n) = dv_0 \wedge v_1 \wedge \cdots \wedge v_n - v_0 \wedge d(v_1 \wedge \cdots \wedge v_n). \quad (24)$$

Explicitly,

$$d\alpha(x_0, \dots, x_n) = \sum_{i < j} (-1)^{i+j} \alpha([x_i, x_j], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n). \quad (25)$$

If V is a Lie algebra, this derivation is a *homological* operator, $d^2 = 0$, called the *Chevalley-Eilenberg cohomology operator* and defining the *Lie algebra cohomology* in the standard way: $H^\bullet(V) = (\text{Ker } d / \text{Im } d)^\bullet$. The cohomology operator d contains the full information about the Lie algebra structure, as the bracket is its dual d^* . We can therefore formulate an equivalent definition of a finite-dimensional Lie algebra as follows.

Proposition 2.10. *A Lie algebra structure on a finite-dimensional vector space V is a degree 1 derivation of the Grassmann algebra $\mathbb{A}(V^*)$ which is homological, i.e. $d^2 = 0$.*

We will formulate later a similar fact for Lie algebroids understanding, after Vaintrob [97, 98], the corresponding derivations as *homological vector fields*.

2.4 Gerstenhaber and Nijenhuis-Richardson brackets

We denote with $M^p(V)$ the space of all p -linear maps $A : V^p \rightarrow V$ if $p > 0$. We put $M^0(V) = V$ and we set $M(V) = \bigoplus_{p \geq 0} M^p(V)$. On the graded vector space $M(V)$ we define the operation $i : M(V)^2 \rightarrow M(V)$ of degree -1 by: $i(B)A = 0$ if $A \in M^0(V)$, and

$$i(B)A(x_1, \dots, x_{a+b-1}) = \sum_{k=1}^a (-1)^{(k-1)(b-1)} A(x_1, \dots, x_{k-1}, B(x_k, \dots, x_{k+b-1}), x_{k+b}, \dots, x_{a+b-1}) \quad (26)$$

if $A \in M^a(V)$, $a > 0$, and $B \in M^b(V)$. Define now the bracket $[\cdot, \cdot]^G : M(V)^2 \rightarrow M(V)$ of degree -1 by

$$[A, B]^G = i(B)A - (-1)^{(a-1)(b-1)} i(A)B, \quad A \in M^a(V), \quad B \in M^b(V). \quad (27)$$

This bracket is an extension of the usual commutator bracket in $M^1(V) = \mathfrak{gl}(V)$, called the *Gerstenhaber bracket* [18].

For the graded subspace $\text{Al}(V)$ of $M(V)$ of alternating (skew-symmetric) mappings, define the bracket $[\cdot, \cdot]^{RN} : \text{Al}(V)^2 \rightarrow \text{Al}(V)$ of degree -1 by

$$[A, B]^{NR} = \frac{(a+b-1)!}{a!b!} \sigma([A, B]^G), \quad A \in \text{Al}^a(V), \quad B \in \text{Al}^b(V), \quad (28)$$

where σ stands for the anti-symmetrization projector in $M(V)$. This bracket is called the *(algebraic) Nijenhuis-Richardson bracket*. The importance of the above brackets indicates the following observation which shows that they serve for determining associative and Lie algebra structures, together with the corresponding cohomology.

Proposition 2.11. *The brackets $[\cdot, \cdot]^G$ and $[\cdot, \cdot]^{NR}$ are graded Lie brackets of degree -1 on $M(V)$ and $\text{Al}(V)$, respectively. Moreover, a map $A \in M^2(V)$ (resp., $A \in \text{Al}^2(V)$) defines an associative (resp., Lie) algebra structure on V if and only if A is a homological element, i.e. $[A, A]^G = 0$ (resp., $[A, A]^{NR} = 0$). In this case, the adjoint map $\partial_A : M(V) \rightarrow M(V)$, $\partial_A(B) = [B, A]^G$ (resp., $\partial_A : \text{Al}(V) \rightarrow \text{Al}(V)$, $\partial_A(B) = [B, A]^{NR}$) is homogeneous of degree 1 and satisfies $\partial_A^2 = 0$, so that it defines a cohomology, called the Hochschild (resp., Chevalley-Eilenberg) cohomology.*

2.5 Poisson brackets

If we have an isomorphism of the vector bundles T^*M and TM , thus inducing an isomorphism $\Omega^1(M) \ni \alpha \mapsto \hat{\alpha} \in \mathcal{X}(M)$, then we can transform the Lie bracket of vector fields into the space $\Omega^1(M)$ of one-forms such that

$$[\alpha, \beta]^\wedge = [\hat{\alpha}, \hat{\beta}]. \quad (29)$$

For instance, a symplectic form ω on M induces such an isomorphism, $\tilde{\omega} : TM \rightarrow T^*M$, and the corresponding bracket $[\alpha, \beta]_\omega$ via

$$\omega(\cdot, \hat{\alpha}) = \alpha. \quad (30)$$

As easy calculations show, for any one-forms α, β and any vector field X ,

$$0 = d\omega(\hat{\alpha}, \hat{\beta}, X) = -d\alpha(\hat{\beta}, X) + d\beta(\hat{\alpha}, X) - \omega([\hat{\beta}, \hat{\alpha}], X) + i_X d(\omega(\hat{\alpha}, \hat{\beta})), \quad (31)$$

so that

$$\begin{aligned} [\alpha, \beta]_\omega &= i_{\hat{\alpha}} d\beta - i_{\hat{\beta}} d\alpha + d(\omega(\hat{\alpha}, \hat{\beta})) \\ &= \mathcal{L}_{\hat{\alpha}} \beta - \mathcal{L}_{\hat{\beta}} \alpha - d(\omega(\hat{\alpha}, \hat{\beta})), \end{aligned} \quad (32)$$

where \mathcal{L} denotes the Lie derivative. The bracket (32) is called the *Koszul bracket* of one-forms. If α and β are exact, $\alpha = df$ and $\beta = dg$, the vector fields $X_f = \widehat{df}$ and $X_g = \widehat{dg}$ are called the *Hamiltonian vector fields* with *Hamiltonians* f and g , respectively. In this case, we have

$$[df, dg]_\omega = d(\omega(X_f, X_g)) = d(X_f(g)) \quad (33)$$

and

$$[X_f, X_g] = [\widehat{df}, \widehat{dg}] = \widehat{[df, dg]} = d(\omega(X_f, X_g))^\wedge, \quad (34)$$

so that the de Rham derivative is a homomorphism of the bracket

$$\{f, g\}_\omega = \omega(X_f, X_g) = X_f(g) \quad (35)$$

on $C^\infty(M)$ into the Koszul bracket,

$$d\{f, g\}_\omega = [df, dg]_\omega. \quad (36)$$

Actually, $\{f, g\}_\omega$ is a Lie bracket:

$$\begin{aligned} \{\{f, g\}_\omega, h\}_\omega &= X_{\{f, g\}_\omega}(h) = [X_f, X_g](h) = X_f(X_g(h)) - X_g(X_f(h)) \\ &= \{f, \{g, h\}_\omega\}_\omega - \{g, \{f, h\}_\omega\}_\omega. \end{aligned} \quad (37)$$

Definition 2.12. A Lie bracket $[\cdot, \cdot]$ on an associative algebra (V, \circ) such that the operators ad_x act as derivations also for the associative multiplication, i.e. the Leibniz rule

$$[x, y \circ z] = [x, y] \circ z + y \circ [x, z] \quad (38)$$

is satisfied, is called a *Poisson bracket*, and the triple $(V, \circ, [\cdot, \cdot])$ a *Poisson algebra*.

Note that any associative algebra is automatically a Poisson algebra with respect to the commutator bracket. Of course, this bracket is trivial for any commutative algebra, so Poisson brackets are extra structures for the latter.

Example 2.13. If (M, ω) is a symplectic manifold, then the bracket (35) is a Poisson bracket and turns $C^\infty(M)$ into a Poisson algebra. Indeed,

$$\{f, gh\}_\omega = X_f(gh) = X_f(g)h + gX_f(h) = \{f, g\}_\omega h + g\{f, h\}_\omega. \quad (39)$$

Due to the Leibniz rule (38), any Poisson bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$ is represented by a bivector field Λ , so that

$$\{f, g\} = \{f, g\}_\Lambda = \langle \Lambda, df \wedge dg \rangle. \quad (40)$$

This is the contravariant version of (35). Of course, in view of the Jacobi identity, the tensor Λ must satisfy an additional condition. In local coordinates, if

$$\Lambda = \frac{1}{2} \Lambda^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \quad (41)$$

then

$$\{f, g\}_\Lambda = \Lambda^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \quad (42)$$

is a Poisson bracket if and only if, for all j, k, l ,

$$\sum_i (\Lambda^{ij} \frac{\partial \Lambda^{kl}}{\partial x^i} + \Lambda^{ik} \frac{\partial \Lambda^{lj}}{\partial x^i} + \Lambda^{il} \frac{\partial \Lambda^{jk}}{\partial x^i}) = 0. \quad (43)$$

Such tensors Λ we call *Poisson tensors* or *Poisson structures*. The above conditions have a nice interpretation in terms of the so called *Schouten-Nijenhuis bracket* (see the next paragraph).

Remark 2.14. It can be proven that the skew-symmetry of Poisson brackets on $C^\infty(M)$ follows from the Leibniz rule and the Jacobi identity [34], so it is a superfluous condition in the definition. In [24], a canonical extension of the Poisson bracket of functions to a graded Lie bracket on differential forms has been constructed. This bracket, however is not a graded Poisson bracket, as the Leibniz rule is not satisfied. Actually, it is a second-order bracket.

Example 2.15. (KKS-structure) Let \mathfrak{g} be a finite-dimensional real Lie algebra with a Lie bracket $[\cdot, \cdot]$ and let c_{ij}^k be the structure constants with respect to a basis x_1, \dots, x_n . Note that x_1, \dots, x_n can be viewed as linear functions defining a coordinate system on the dual space \mathfrak{g}^* . Then, there is a uniquely determined Poisson bracket $\{\cdot, \cdot\}$ on \mathfrak{g}^* (*Kostant-Kirillov-Souriau bracket*) such that

$$\{x_i, x_j\} = [x_i, x_j] = c_{ij}^k x_k. \quad (44)$$

Indeed, it is easy to see that the corresponding tensor must be $\Lambda = \frac{1}{2} c_{ij}^k x_k \partial_{x_i} \wedge \partial_{x_j}$ which satisfies (43), as the latter is in this case equivalent to the Jacobi identity for $[\cdot, \cdot]$. The Poisson tensor is *linear* in the obvious sense and hence the corresponding Poisson bracket is closed on polynomial functions.

One important observation is that the above correspondence between Lie brackets on a vector space and linear Poisson tensors on the dual remains valid for an arbitrary vector bundle $\tau : E \rightarrow M$ (see Theorem 3.1). Of course, linear functions on a vector bundle are understood as functions which are linear along fibres, and the linearity of a Poisson tensor means that the corresponding Poisson bracket is closed on linear functions. Automatically, it is closed on the space of polynomial functions which becomes, in this way, a Lie algebra.

Example 2.16. It is well known that the cotangent bundle \mathbb{T}^*M possesses a canonical symplectic structure ω_M , thus $C^\infty(\mathbb{T}^*M)$ is canonically a Poisson algebra. There are local affine coordinates (q^a, p_a) on \mathbb{T}^*M , called *Darboux coordinates*, in which $\omega_M = dp_a \wedge dq^a$ and in which this Poisson bracket reads as

$$\{f, g\} = \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q^a} - \frac{\partial f}{\partial q^a} \frac{\partial g}{\partial p_a}. \quad (45)$$

The corresponding Poisson tensor $\Lambda_M = \partial_{p_a} \wedge \partial_{q^a}$ is the ‘inverse’ of the symplectic form ω_M in the sense that, *via* the contraction, it defines the inverse isomorphism

$$\widetilde{\Lambda}_M : \mathbb{T}^*\mathbb{T}^*M \rightarrow \mathbb{T}\mathbb{T}^*M. \quad (46)$$

The tensor is linear, because the bracket is closed on linear (i.e. linear in p ’s) functions. Consequently, polynomial (in p ’s) functions form a Lie algebra. This is the Lie algebra of symbols of differential operators on M .

Theorem 2.1. (*Darboux Theorem*) *Each symplectic Poisson bracket can be written locally in the form (45).*

Let us observe that the linear Poisson bracket (45) is *de facto* equivalent to the Lie bracket of vector fields on M . Indeed, we can identify any vector field X on M with the corresponding linear function ι_X on \mathbb{T}^*M in an obvious way: $\iota(X)(\alpha_q) = \langle X(q), \alpha_q \rangle$. In local coordinates,

$$\iota(f^a(q)\partial_{q^a}) = f^a(q)p_a. \quad (47)$$

It is easy to see now that

$$\{\iota(X), \iota(Y)\} = \iota([X, Y]), \quad (48)$$

$$\{\iota(X), f\} = X(f), \quad (49)$$

where f is any basic function on \mathbb{T}^*M interpreted as a function on M . A more detailed study of Poisson brackets and related structures can be found in [96].

2.6 Jacobi and graded Jacobi brackets

A construction of a Lie bracket, similar to that for functions on a symplectic manifold, can be done in the case of a contact manifold (M, α) . We call this bracket the *Legendre bracket*.

Example 2.17. Not going into a general theory, let us recall that any contact form can be locally written as $\alpha = dz - p_a dq^a$ (Darboux theorem) and the Legendre bracket of functions f, g on M in these coordinates reads as

$$\{f, g\}_\alpha = \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q^a} - \frac{\partial f}{\partial q^a} \frac{\partial g}{\partial p_a} + \frac{\partial f}{\partial z} \left(g - p_a \frac{\partial g}{\partial p_a} \right) - \left(f - \frac{\partial f}{\partial p_a} p_a \right) \frac{\partial g}{\partial z}. \quad (50)$$

This bracket is not Poisson, since the Leibniz rule is not satisfied: the operators $\{f, \cdot\}_\alpha$ act on $C^\infty(M)$ as first-order differential operators, not derivations. This can be expressed in terms of a *generalized Leibniz rule*:

$$\{f, gh\}_\alpha = \{f, g\}_\alpha h + g\{f, h\}_\alpha - \{f, \mathbf{1}\}_\alpha gh. \quad (51)$$

A Lie bracket on a (commutative) associative unital algebra, satisfying (51), we call a *Jacobi bracket*. Thus the bracket (50) is an example of a Jacobi bracket on $C^\infty(M)$. In general, Jacobi brackets on $C^\infty(M)$ are represented by pairs (Λ, Γ) , where Λ is a bivector field and Γ is a vector field, by

$$\{f, g\}_{(\Lambda, \Gamma)} = \Lambda(df, dg) + \Gamma(f)g - f\Gamma(g). \quad (52)$$

The pair (Λ, Γ) is called a *Jacobi structure* [62]. For the Legendre bracket (50),

$$\Lambda = \partial_{p_a} \wedge \partial_{q^a} + p_a \partial_{p_a} \wedge \partial_z, \quad \Gamma = \partial_z, \quad (53)$$

Of course, the Jacobi identity for the bracket implies some compatibility conditions for Λ and Γ (cf. (59)). Poisson bracket are just Jacobi brackets with $\Gamma = 0$, i.e. such that $\mathbf{1}$ is a central element, $\{\mathbf{1}, \cdot\} = 0$. The concepts of a Poisson and a Jacobi bracket can be easily extended to the graded case.

Definition 2.18. A *graded Jacobi bracket* of degree k on a G -graded (think e.g. \mathbb{Z} -graded) associative commutative algebra $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}^g$ with unity $\mathbf{1}$ is a graded bilinear map

$$\{\cdot, \cdot\} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} \quad (54)$$

of degree k , i.e. $w(\{a, b\}) = w(a) + w(b) + k$, such that

1. $\{a, b\} = -(-1)^{\langle w(a)+k, w(b)+k \rangle} \{b, a\}$ (graded anticommutativity),
2. $\{a, bc\} = \{a, b\}c + (-1)^{\langle w(a)+k, b \rangle} b\{a, c\} - \{a, \mathbf{1}\}bc$ (generalized graded Leibniz rule),
3. $\{\{a, b\}, c\} = \{a, \{b, c\}\} - (-1)^{\langle w(a)+k, w(b)+k \rangle} \{b, \{a, c\}\}$ (graded Jacobi identity).

Such a bracket is called a *graded Poisson bracket* if $\mathbf{1}$ is its central element. Note that \mathbb{Z} -graded algebras furnished with a graded Poisson bracket of degree -1 are sometimes called *Gerstenhaber algebras* (see [54], [55]).

Definition 2.19. If H is an element in a graded Poisson algebra \mathcal{A} , we can consider the bracket $\{a, b\}^H = \{\{a, H\}, b\}$, called the *derived bracket* (associated with the ‘Hamiltonian’ H) [56]. The odd Hamiltonians we call *homological* if $\{H, H\} = 0$ (the latter condition is nontrivial for odd Hamiltonians).

Problem. Show that, if the graded Poisson bracket is of degree k and H is a homological Hamiltonian of degree h , then the derived bracket is a graded Lie bracket of degree $h + 2k$.

Example 2.20. (The Schouten bracket) Several natural graded Lie brackets of tensor fields are associated with a given smooth (C^∞) manifold M . Historically the first one was probably the celebrated Schouten-Nijenhuis bracket $[\cdot, \cdot]^{SN}$ defined on multivector fields (see [81, 88] for the original and [78] for a modern version). It is the unique graded Poisson extension of the usual bracket $[\cdot, \cdot]$ of vector fields to the Grassmann algebra $\mathcal{X}(M) = \bigoplus_{n \in \mathbb{N}} \mathcal{X}^n(M)$ of multivector fields. Consequently,

- the degree of $X \in \mathcal{X}^n(M)$ with respect to the bracket is $(n - 1)$,
- $[X, f]^{SN} = X(f)$ for $X \in \mathcal{X}^1(M)$, $f \in \mathcal{X}^0(M) = C^\infty(M)$;
- For $X \in \mathcal{X}^k(M)$, $Y \in \mathcal{X}^l(M)$, we have

$$[X, Y \wedge Z]^{SN} = [X, Y]^{SN} \wedge Z + (-1)^{(k-1)l} Y \wedge [X, Z]^{SN}. \quad (55)$$

Explicitly,

$$[X_1 \wedge \dots \wedge X_r, Y_1 \wedge \dots \wedge Y_n]^{SN} = \sum_{k,l} (-1)^{k+l} [X_k, Y_l] \wedge \dots \wedge \widehat{X_k} \wedge \dots \wedge X_r \wedge Y_1 \wedge \dots \wedge \widehat{Y_l} \wedge \dots \wedge Y_n, \quad (56)$$

where $X_k, Y_l \in \mathcal{X}^1(M)$ and ‘ \wedge ’ stand for the omission. Note that $[X, Y]^{SN}$, with X being a vector field, is just the Lie derivative $\mathcal{L}_X Y$.

It is easy to see that condition (43) defining a Poisson tensor can be rewritten in terms of the Schouten-Nijenhuis bracket as

$$[\Lambda, \Lambda]^{SN} = 0. \quad (57)$$

Moreover, the corresponding Poisson bracket can be viewed as the derived bracket:

$$\{f, g\}_\Lambda = [[f, -\Lambda]^{SN}, g]^{SN}. \quad (58)$$

The Poisson tensor $-\Lambda$ plays the role of the homological Hamiltonian which is quadratic (of degree 2). The derived bracket is therefore of degree $2 + 2(-1) = 0$, so closed on basic (degree 0) functions.

Similarly to (57), (52) is a Jacobi bracket if and only if

$$[\Lambda, \Lambda]^{SN} = 2\Lambda \wedge \Gamma \quad \text{and} \quad [\Gamma, \Lambda]^{SN} = \mathcal{L}_\Gamma \Lambda = 0. \quad (59)$$

Remark 2.21. One can consider as well the *symmetric Schouten bracket* (see [12]). It is an ordinary (non-graded) Lie bracket extending the Lie bracket of vector fields, defined on symmetric contravariant tensors and satisfying an analog of (56):

$$[X_1 \vee \dots \vee X_r, Y_1 \vee \dots \vee Y_n]^{SS} = \sum_{k,l} [X_k, Y_l] \vee \dots \vee \widehat{X_k} \vee \dots \wedge X_r \wedge Y_1 \wedge \dots \wedge \widehat{Y_l} \wedge \dots \wedge Y_n. \quad (60)$$

The symmetric Schouten bracket is, however, nothing but the standard symplectic Poisson bracket on \mathbb{T}^*M reduced to polynomial functions. Polynomial functions on \mathbb{T}^*M represent, namely, symmetric contravariant tensors by an extension of (47),

$$\iota(f(q)\partial_{q^{a_1}} \vee \dots \vee \partial_{q^{a_n}}) = f(q)p_{a_1} \cdots p_{a_n}, \quad (61)$$

and the brackets are identified according to (48) and (49).

3 Algebroids

Lie algebroids are geometric objects which are so common and natural that we are often working with them not even mentioning it. The people told that they are using a Lie algebroid resemble Mr. Jourdain who was surprised and delighted to learn that he has been speaking prose all his life without knowing it. One can consider also a more general object, a *skew algebroid*, for which we drop the Jacobi identity.

A *Lie pseudotalgebra*, a pure algebraic counterpart of a Lie algebroid, appeared first in the paper of Herz [45] but one can find similar concepts under more than a dozen of names in the literature (e.g. Lie modules, (R, A) -Lie algebras, Lie-Cartan pairs, Lie-Rinehart algebras, differential algebras, etc.). Lie algebroids were introduced by Pradines [85] as infinitesimal parts of differentiable groupoids. In the same year the booklet [80] by Nelson was published, where a general theory of Lie modules together with a big part of the corresponding differential calculus can be found. We also refer to a survey article by Mackenzie [67] and his book [68].

3.1 Skew algebroids

Let $\tau : E \rightarrow M$ be a rank- n vector bundle over an m -dimensional manifold M and let $\pi : E^* \rightarrow M$ be its dual. Recall that the Grassmann algebra $\mathbb{A}(E) = \bigoplus_{i=0}^{\infty} \text{Sec}(\wedge^i E)$ of multisections of E is a graded commutative associative algebra with respect to the wedge product.

There are different equivalent ways to define a *skew algebroid* structure on E . Here we will list only four of them. The notation is borrowed from [19, 21, 42] and we refer to these papers for details. In particular, we use affine coordinates (x^a, ξ_i) on E^* and the dual coordinates (x^a, y^i) on E , associated with dual local bases, (e_i) and (e^i) , of sections of E and E^* , respectively.

Definition 3.1. A *skew algebroid* structure on E is given by a linear bivector field Π on E^* . In local coordinates,

$$\Pi = \frac{1}{2} c_{ij}^k(x) \xi_k \partial_{\xi_i} \wedge \partial_{\xi_j} + \rho_i^b(x) \partial_{\xi_i} \wedge \partial_{x^b}, \quad (62)$$

where $c_{ij}^k(x) = -c_{ji}^k(x)$. If Π is a Poisson tensor, we speak about a *Lie algebroid*.

As the bivector field Π defines a bilinear bracket $\{\cdot, \cdot\}^\Pi$ on the algebra $C^\infty(E^*)$ of smooth functions on E^* by $\{\phi, \psi\}^\Pi = \langle \Pi, d\phi \wedge d\psi \rangle$, where $\langle \cdot, \cdot \rangle$ stands for the contraction, we get the following.

Theorem 3.1. A *skew algebroid* structure (E, Π) can be equivalently defined as

- a skew-symmetric \mathbb{R} -bilinear bracket $[\cdot, \cdot]^\Pi$ on the space $\text{Sec}(E)$ of sections of E , together with a vector bundle morphisms $\rho = \rho^\Pi : E \rightarrow TM$ (the anchor), such that

$$[X, fY]^\Pi = \rho^\Pi(X)(f)Y + f[X, Y]^\Pi, \quad (63)$$

for all $f \in C^\infty(M)$, $X, Y \in \text{Sec}(E)$;

- a graded skew-symmetric bracket $[[\cdot, \cdot]]^\Pi$ of degree -1 , the algebroid Schouten bracket, on the Grassmann algebra $\mathbb{A}(E)$, satisfying the Leibniz rule

$$[[X, Y \wedge Z]]^\Pi = [[X, Y]]^\Pi \wedge Z + (-1)^{(k-1)l} Y \wedge [[X, Z]]^\Pi, \quad (64)$$

for $X \in \mathbb{A}^k(E)$, $Y \in \mathbb{A}^l(E)$;

- or as a graded derivation d^Π of degree 1 in the Grassmann algebra $\mathbb{A}(E^*)$ (the de Rham derivative),

$$d^\Pi(\alpha \wedge \beta) = d^\Pi \alpha \wedge \beta + (-1)^{w(\alpha)} \alpha \wedge d^\Pi \beta. \quad (65)$$

Moreover, the following properties of the above structures are equivalent:

- (E, Π) is a Lie algebroid.
- $[\cdot, \cdot]^\Pi$ is a Lie bracket.
- $[[\cdot, \cdot]]^\Pi$ is a graded Poisson bracket.
- $(d^\Pi)^2 = 0$.

In the latter case, the Lie algebroid cohomology is defined in the standar way: $H^\bullet(E; d^\Pi) = (\text{Ker } d^\Pi / \text{Im } d^\Pi)^\bullet$.

The bracket $[\cdot, \cdot]^\Pi$ and the anchor ρ^Π are related to the bracket $\{\cdot, \cdot\}^\Pi$ according to the formulae:

$$\iota([X, Y]^\Pi) = \{\iota(X), \iota(Y)\}^\Pi, \quad (66)$$

$$\pi^*(\rho^\Pi(X)(f)) = \{\iota(X), \pi^*f\}^\Pi, \quad (67)$$

where we denoted with $\iota(X)$ the linear function on E^* associated with the section X of E , i.e. $\iota(X)(e_p^*) = \langle X(p), e_p^* \rangle$ for each $e_p^* \in E_p^*$.

The algebroid Schouten bracket is the unique graded extension of $[\cdot, \cdot]^\Pi$ satisfying the Leibniz rule. The *de Rham derivative* d^Π is determined by the formula

$$(d^\Pi \mu)^v = [\Pi, \mu^v]^{SN}, \quad (68)$$

where μ^v is the natural vertical lift of a ' k -form' $\mu \in \mathbb{A}^k(E^*)$ to a vertical k -vector field on E^* and $[\cdot, \cdot]^{SN}$ is the Schouten-Nijenhuis bracket of multivector fields. It can also be written in the *Cartan form*

$$\begin{aligned} d^\Pi \mu(X_1, \dots, X_{k+1}) &= \sum_i (-1)^{i+1} \rho^\Pi(X_i)(\mu(X_1, \dots, \hat{X}_i, \dots, X_{k+1})) \\ &+ \sum_{i < j} (-1)^{i+j} \mu([X_i, X_j]^\Pi, X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}). \end{aligned} \quad (69)$$

In local bases of sections and the corresponding local coordinates,

$$[e_i, e_j]^\Pi(x) = c_{ij}^k(x) e_k, \quad (70)$$

$$\rho^\Pi(e_i)(x) = \rho_i^a(x) \partial_{x^a}, \quad (71)$$

$$(d^\Pi f)(x) = \rho_i^a(x) \frac{\partial f}{\partial x^a}(x) e^i, \quad (72)$$

$$(d^\Pi e^i)(x) = c_{lk}^i(x) e^k \wedge e^l. \quad (73)$$

Given a skew algebroid E , we can associate with any C^1 -function H on E^* its *Hamiltonian vector field* \mathcal{X}_H like in the standard case: $\mathcal{X}_H = i_{dH} \Pi$ that allows for a sort of 'Hamiltonian mechanics'. In local coordinates,

$$\mathcal{X}_H(x, \xi) = \left(c_{ij}^k(x) \xi_k \frac{\partial H}{\partial \xi_i}(x, \xi) - \rho_j^a(x) \frac{\partial H}{\partial x^a}(x, \xi) \right) \partial_{\xi_j} + \rho_i^b(x) \frac{\partial H}{\partial \xi_i}(x, \xi) \partial_{x^b}. \quad (74)$$

Another geometrical construction in the skew-algebroid setting is the *complete lift of an algebroid section* (cf. [41, 42]). For every C^1 -section, $X = f^i(x) e_i \in \text{Sec}(E)$, we can construct canonically a vector field $d_T^\Pi(X) \in \text{Sec}(TE)$ which in local coordinates reads as

$$d_T^\Pi(X)(x, y) = f^i(x) \rho_i^a(x) \partial_{x^a} + \left(y^i \rho_i^a(x) \frac{\partial f^k}{\partial x^a}(x) + c_{ij}^k(x) y^i f^j(x) \right) \partial_{y^k}. \quad (75)$$

The vector field $d_T^\Pi(X)$ is homogeneous (linear with respect to y 's).

Theorem 3.2 ([41, 42]). *The pair (E, Π) defines a Lie algebroid if and only if $d_T^\Pi([X, Y]^\Pi) = [d_T^\Pi(X), d_T^\Pi(Y)]$ for all $X, Y \in \text{Sec}(E)$.*

Example 3.2. Any tangent bundle $E = TM$ of a manifold M , with $\rho = \text{Id}_{TM}$ and the usual Lie bracket of vector fields, is a Lie algebroid.

Example 3.3. Any Lie algebra, $E = \mathfrak{g}$, considered as a vector bundle over one point $M = \{pt\}$ with the trivial anchor $\rho = 0$, is a Lie algebroid. This Lie algebroid can be viewed as a reduction of the tangent bundle of any Lie group G associated with \mathfrak{g} , namely $\mathfrak{g} = \mathfrak{T}G/G$, in which sections of \mathfrak{g} are interpreted as invariant vector fields on G .

Example 3.4. The above reduction procedure can be generalized to the case of any principal G -bundle P . Invariant vector fields on P are closed with respect to the Lie bracket and can be viewed as sections of the vector bundle $E = \mathfrak{T}P/G$ which becomes a Lie algebroid. This is the so called *Atiyah algebroid* associated with the principal bundle P .

Example 3.5. There is a canonical Lie algebroid structure on the cotangent bundle \mathfrak{T}^*M associated with a Poisson tensor Λ on M . This is the unique Lie algebroid bracket $[\cdot, \cdot]_\Lambda$ of differential 1-forms for which $[df, dg]_\Lambda = d\{f, g\}_\Lambda$, where $\{\cdot, \cdot\}_\Lambda$ is the Poisson bracket of functions for Λ and the anchor map is $\tilde{\Lambda} : \mathfrak{T}^*M \rightarrow \mathfrak{T}M$. Explicitly (cf. (32)),

$$[\alpha, \beta]_\Lambda = \mathcal{L}_{\tilde{\Lambda}(\alpha)}\beta - \mathcal{L}_{\tilde{\Lambda}(\beta)}\alpha - d\langle \Lambda, \alpha \wedge \beta \rangle. \quad (76)$$

This Lie bracket was defined first by Fuchssteiner [16] but it is usually called the *Koszul bracket* [60]. The corresponding linear Poisson tensor on $\mathfrak{T}M$ is the tangent lift $d_\mathfrak{T}\Lambda$ [41, 42]. As the tangent lift respects the Schouten bracket [39], it is again a Poisson structure, this time linear.

In the next sections we will show two more interpretations of a skew algebroid: as a morphism of *double vector bundles* and as a vector field on a *graded manifold*.

3.2 Differential calculus on Lie algebroids

Let us consider a Lie algebroid structure (E, Π) on a vector bundle $\tau : E \rightarrow M$, with the Lie bracket $[\cdot, \cdot]^\Pi$ on sections of E and the anchor $\rho^\Pi : E \rightarrow \mathfrak{T}M$. Then, we can construct a well-known generalization of the standard Cartan calculus of differential forms and vector fields (see e.g. [68, 73]).

First, we have the exterior (de Rham) derivative $d^\Pi : \mathbb{A}^k(E^*) \rightarrow \mathbb{A}^{k+1}(E^*)$ (69). For $X \in \mathbb{A}^k(E)$, the contraction $i_X : \mathbb{A}^p(E^*) \rightarrow \mathbb{A}^{p-k}(E^*)$ is defined in the standard way for $k = 1$ and extended by $i_{X_1 \wedge \dots \wedge X_k} = i_{X_1} \cdots i_{X_k}$ for $X_i \in \text{Sec}(E)$ (this produces a sign factor with respect to another convention for the contraction). The Lie differential operator

$$\mathcal{L}_X^\Pi : \mathbb{A}^p(E^*) \rightarrow \mathbb{A}^{p-k+1}(E^*) \quad (77)$$

is defined as the graded commutator

$$\mathcal{L}_X^\Pi = [i_X, d^\Pi] = i_X \circ d^\Pi - (-1)^k d^\Pi \circ i_X. \quad (78)$$

The following proposition contains a list of well-known properties of these objects.

Proposition 3.6. *Let $\mu \in \mathbb{A}^k(E^*)$, $\nu \in \mathbb{A}(E^*)$ and $X, Y \in \mathbb{A}^1(E)$. We have*

1. $d^\Pi \circ d^\Pi = 0$,
2. $d^\Pi(\mu \wedge \nu) = d^\Pi \mu \wedge \nu + (-1)^k \mu \wedge d^\Pi \nu$,
3. $i_X(\mu \wedge \nu) = i_X \mu \wedge \nu + (-1)^k \mu \wedge i_X \nu$,
4. $\mathcal{L}_X^\Pi(\mu \wedge \nu) = \mathcal{L}_X^\Pi \mu \wedge \nu + \mu \wedge \mathcal{L}_X^\Pi \nu$,

$$5. [\mathcal{L}_X^\Pi, \mathcal{L}_Y^\Pi] = \mathcal{L}_X^\Pi \circ \mathcal{L}_Y^\Pi - \mathcal{L}_Y^\Pi \circ \mathcal{L}_X^\Pi = \mathcal{L}_{[X,Y]^\Pi}^\Pi,$$

$$6. [\mathcal{L}_X^\Pi, i_Y] = \mathcal{L}_X^\Pi \circ i_Y - i_Y \circ \mathcal{L}_X^\Pi = i_{[X,Y]^\Pi}.$$

The last formula can be generalized in the following way (cf. [40, 73, 78]).

Theorem 3.3. For $X \in \mathbb{A}^{k+1}(E)$ and $Y \in \mathbb{A}^{l+1}(E)$,

$$[\mathcal{L}_X^\Pi, i_Y] = \mathcal{L}_X^\Pi \circ i_Y - (-1)^{(k+1)l} i_Y \circ \mathcal{L}_X^\Pi = i_{[X,Y]^\Pi}, \quad (79)$$

where $[\cdot, \cdot]^\Pi$ is the algebroid Schouten bracket. In particular, for $X \in \mathbb{A}^1(E)$ and $f \in \mathbb{A}^0(E) = C^\infty(M)$ we have

$$[X, f]^\Pi = \rho^\Pi(X)(f). \quad (80)$$

There is also a *symmetric Schouten bracket* which extends the Schouten bracket on $\mathbb{A}^0(E) \oplus \mathbb{A}^1(E)$ to symmetric multisections. This bracket is just the polynomial part of the Poisson bracket $\{\cdot, \cdot\}^\Pi$. Before we pass to other brackets, let us introduce the bi-graded space

$$\Phi(E) = \bigoplus_{k,l=0}^\infty \Phi_l^k(E) = \mathbb{A}(E \oplus_M E^*), \quad \Phi_l^k(E) = \text{Sec}(\wedge^k E \otimes_M \wedge^l E^*), \quad (81)$$

of tensor fields of mixed type. Of course, we can identify $\Phi_l^k(E)$ and $\Phi_k^l(E^*)$. For $K \in \Phi_1^k(E^*)$, we define the contraction

$$i_K: \mathbb{A}^n(E^*) \rightarrow \mathbb{A}^{n+k-1}(E^*) \quad (82)$$

in a natural way: for simple tensors $K = \mu \otimes X$, where $\mu \in \mathbb{A}^k(E^*)$, $X \in \mathbb{A}^1(E)$, we just put

$$i_K \nu = \mu \wedge i_X \nu. \quad (83)$$

The corresponding Lie differential is defined by the formula

$$\mathcal{L}_K^\Pi = i_K \circ d^\Pi + (-1)^k d^\Pi \circ i_K \quad (84)$$

and, in particular,

$$\mathcal{L}_{\mu \otimes X}^\Pi = \mu \wedge \mathcal{L}_X^\Pi + (-1)^k d^\Pi \mu \wedge i_X. \quad (85)$$

This definition is compatible with the previous one in the case of $K \in \Phi_1^0(E^*) = \mathbb{A}^1(E) = \text{Sec}(E)$. The contraction (insertion) i_K can be extended to an operator

$$i_K: \Phi_1^n(E^*) \rightarrow \Phi_1^{n+k-1}(E^*) \quad (86)$$

by the formula

$$i_K(\mu \otimes X) = i_K(\mu) \otimes X. \quad (87)$$

Theorem 3.4. The bracket

$$[\cdot, \cdot]^{NR}: \Phi_1^{k+1}(E^*) \times \Phi_1^{l+1}(E^*) \rightarrow \Phi_1^{k+l+1}(E^*), \quad (88)$$

given by the formula

$$[K, L]^{NR} = i_K L - (-1)^{kl} i_L K, \quad (89)$$

defines a graded Lie algebra structure on the graded space $\Phi_1(E^*) = \bigoplus_{k \in \mathbb{N}} \Phi_1^k(E^*)$. For simple tensors $\mu \otimes X \in \Phi_1^k(E^*)$ and $\nu \otimes Y \in \Phi_1^l(E^*)$, we get

$$[\mu \otimes X, \nu \otimes Y]^{NR} = \mu \wedge i_X \nu \otimes Y + (-1)^k i_Y \mu \wedge \nu \otimes X. \quad (90)$$

The bracket $[\cdot, \cdot]^{NR}$ is called the (generalized) *Nijenhuis-Richardson bracket*.

Remark 3.7. The generalized Nijenhuis-Richardson bracket is a purely vector bundle bracket and does not depend on the Lie algebroid structure. It is a geometric counterpart of the purely algebraic bracket (28). For $E = TM$, we get the classical Nijenhuis-Richardson bracket of vector-valued forms [82].

Another important bracket, the *generalized Frölicher-Nijenhuis bracket*, is also a bracket on the graded space $\Phi_1(E^*) = \oplus_{k \in \mathbb{N}} \Phi_1^k(E^*)$ of ‘vector-valued forms’, defined for simple tensors $\mu \otimes X \in \Phi_1^k(E^*)$ and $\nu \otimes Y \in \Phi_1^l(E^*)$ by

$$\begin{aligned} [\mu \otimes X, \nu \otimes Y]_{\Pi}^{FN} &= \mu \wedge \nu \otimes [X, Y]^{\Pi} + \mu \wedge \mathcal{L}_X^{\Pi} \nu \otimes Y - \mathcal{L}_Y^{\Pi} \mu \wedge \nu \otimes X \\ &\quad + (-1)^k (d^{\Pi} \mu \wedge i_X \nu \otimes Y + i_Y \mu \wedge d^{\Pi} \nu \otimes X) \\ &= (\mathcal{L}_{\mu \otimes X}^{\Pi} \nu) \otimes Y - (-1)^{kl} (\mathcal{L}_{\nu \otimes Y}^{\Pi} \mu) \otimes X + \mu \wedge \nu \otimes [X, Y]^{\Pi}. \end{aligned} \quad (91)$$

Theorem 3.5 ([42, 50]). *The formula (91) defines a graded Lie bracket of degree 0 on the graded space $\Phi_1(E^*) = \oplus_{k \in \mathbb{N}} \Phi_1^k(E^*)$ of vector-valued forms. Moreover,*

$$\begin{aligned} [\mathcal{L}_K^{\Pi}, \mathcal{L}_L^{\Pi}] &= \mathcal{L}_K^{\Pi} \circ \mathcal{L}_L^{\Pi} - (-1)^{kl} \mathcal{L}_L^{\Pi} \circ \mathcal{L}_K^{\Pi} = \\ &= \mathcal{L}_{[K, L]_{\Pi}^{FN}}^{\Pi}, \end{aligned} \quad (92)$$

$$\begin{aligned} [\mathcal{L}_K^{\Pi}, i_L] &= \mathcal{L}_K^{\Pi} \circ i_L - (-1)^{k(l+1)} i_L \circ \mathcal{L}_K^{\Pi} \\ &= i_{[K, L]_{\Pi}^{FN}} - (-1)^{k(l+1)} \mathcal{L}_{i_L K}^{\Pi}. \end{aligned} \quad (93)$$

Problem. Prove that, for N being a $(1, 1)$ tensor interpreted as a morphism $N : E \rightarrow E$ of vector bundles, we have

$$[N, N]_{\Pi}^{FN}(X, Y) = [NX, NY]^{\Pi} - N([NX, Y]^{\Pi} + [X, NY]^{\Pi} - N[X, Y]^{\Pi}), \quad (94)$$

for any $X, Y \in \text{Sec}(E)$. The tensor $[N, N]_{\Pi}^{FN}$ is sometimes called the (*generalized*) *Nijenhuis torsion* of N .

In the case of the canonical Lie algebroid $E = TM$, we obtain the classical Frölicher - Nijenhuis bracket on the graded space $\Omega(M; TM) = \Phi_1(T^*M)$ of vector-valued forms [12, 15, 50, 77].

Note that there are some interesting relations of the classical Nijenhuis-Richardson and Frölicher-Nijenhuis brackets on M with the Schouten bracket on T^*M . Let us first recall that any vector field X on M can be identified with a linear function $\iota(X)$ on T^*M . As T^*M is canonically a symplectic manifold, we can associate with $\iota(X)$ its Hamiltonian vector field which will be denoted $\mathcal{G}(X)$ and called the *cotangent lift* of X . Second, any one-form α can be lifted to a vertical vector field $\mathcal{V}(\alpha)$,

$$\mathcal{V}(f_a(q) dq^a) = f_a(q) \partial_{p_a}. \quad (95)$$

This vertical lift can be extended to k -forms by

$$\mathcal{V}(\alpha_1 \wedge \cdots \wedge \alpha_k) = \mathcal{V}(\alpha_1) \wedge \cdots \wedge \mathcal{V}(\alpha_k). \quad (96)$$

We can extend the maps ι and \mathcal{G} to linear maps $\mathcal{J}, \mathcal{G} : \Omega(M; TM) \rightarrow \mathcal{X}(T^*M)$ by

$$\mathcal{J}(\mu \otimes X) = -\iota(X) \mathcal{V}(\mu) \quad (97)$$

and

$$\mathcal{G}(\mu \otimes X) = \mathcal{G}(X) \wedge \mathcal{V}(\mu) - \iota(X) \mathcal{V}(d\mu), \quad (98)$$

for simple tensors $\mu \otimes X \in \Omega(M; TM)$.

Theorem 3.6 ([40]). *The mappings $\mathcal{J}, \mathcal{G} : \Omega(M; TM) \rightarrow \mathcal{X}(\mathbb{T}^*M)$ are injective homomorphisms (embeddings) of, respectively, the Nijenhuis-Richardson and the Frölicher-Nijenhuis bracket into the Schouten-Nijenhuis bracket:*

$$\mathcal{J}([K, L]^{NR}) = [\mathcal{J}(K), \mathcal{J}(L)]^{SN}, \quad (99)$$

$$\mathcal{G}([K, L]^{FN}) = [\mathcal{G}(K), \mathcal{G}(L)]^{SN}. \quad (100)$$

3.3 Nijenhuis tensors

If, for a Lie algebroid (E, Π) , the Nijenhuis torsion (94) of a tensor $N : E \rightarrow E$ vanishes, we call N a *Nijenhuis tensor* (see [58]). The crucial property of a Nijenhuis tensor as defining a contraction of the bracket is the following (cf. [58, 41]).

Theorem 3.7. *If N is a Nijenhuis tensor for a Lie algebroid bracket $[\cdot, \cdot]^\Pi$ on E with an anchor map $\rho^\Pi : E \rightarrow \mathbb{T}M$, then the contracted bracket*

$$[X, Y]_N^\Pi = [NX, Y]^\Pi + [X, NY]^\Pi - N[X, Y]^\Pi \quad (101)$$

is again a Lie algebroid bracket on E with the anchor $\rho_N^\Pi = \rho^\Pi \circ N$. This bracket corresponds to the linear Poisson structure $\Pi_N = \mathcal{L}_{\mathcal{J}(N)}\Pi$. Moreover, $N : E \rightarrow E$ is a morphism of the Lie algebroid (E, Π_N) into the Lie algebroid (E, Π) :

$$[NX, NY]^\Pi = N([X, Y]_N^\Pi). \quad (102)$$

Remark 3.8. If the contracted bracket $[\cdot, \cdot]_N^\Pi$ is again a Lie bracket, then N is called *weak Nijenhuis* (cf. [6]). The above theorem implies that Nijenhuis tensors are weak-Nijenhuis. Tensors $N : \mathbb{T}M \rightarrow \mathbb{T}M$ satisfying $N^2 = -Id$ are called *almost complex structures*. The celebrated Newlander-Nirenberg theorem states that an almost complex structure N is integrable, i.e. comes from a true complex structure, if and only if N is Nijenhuis.

Example 3.9. (Frobenius manifolds) An algebraical part of the structure of a *Frobenius manifold* consists of a unital commutative associative multiplication "o" in the space $\mathcal{X}^1(M)$ of vector fields which comes from a symmetric vector valued two-form $C \in \text{Sec}(\vee^2 \mathbb{T}^*M \otimes_M \mathbb{T}M)$. This multiplication is supposed to satisfy the following axiom proposed by Hertling and Manin:

$$\mathcal{L}_{X \circ Y} C = X \circ \mathcal{L}_Y C + \mathcal{L}_X C \circ Y. \quad (103)$$

In terms of structure functions in local coordinates, (103) reads as

$$C_{sj}^m \frac{\partial C_{lr}^s}{\partial x^k} + C_{sk}^m \frac{\partial C_{lr}^s}{\partial x^j} - C_{sr}^m \frac{\partial C_{jk}^s}{\partial x^l} - C_{sl}^m \frac{\partial C_{jk}^s}{\partial x^r} + \frac{\partial C_{jk}^m}{\partial x^s} C_{lr}^s - \frac{C_{lr}^m}{\partial x^s} C_{jk}^s = 0, \quad (104)$$

for all m, j, k, l, r . An interpretation of the above conditions can be found in an old paper by Yano and Ako [103], where they constructed several classes of "differential concomitants" in the sense of Schouten. One among them leads exactly to (104), so that the above differential constraints on the tensor field C are sometimes referred to as the *Yano-Ako conditions* (see [70] and the discussion there). We will show that (103) can be interpreted as vanishing of a Nijenhuis torsion.

Recall first that symmetric multi-vector fields on the manifold M with the symmetric Schouten bracket can be identified with the graded algebra $\mathcal{A} = \bigoplus_{k=0}^{\infty} \mathcal{A}^k$ of polynomial functions on \mathbb{T}^*M with the symplectic Poisson bracket. Second, the unital commutative

associative multiplication C in $\mathcal{X}^1(M) = \mathcal{A}^1$ defines an $\mathcal{A}^0 = C^\infty(M)$ -linear projection $N = N(C) : \mathcal{A} \rightarrow \mathcal{A}^1$ defined by:

$$N(1) = E_0, \quad N(X_1 \cdots X_k) = X_1 \circ \cdots \circ X_k \quad \text{for all } X_1, \dots, X_k \in \mathcal{A}^1, \quad (105)$$

where $E_0 \in \mathcal{A}^1$ corresponds to the unity vector field of the multiplication. As \mathcal{A}^1 is a Lie subalgebra in \mathcal{A} with the Poisson bracket, N is a Nijenhuis tensor for the Lie algebra structure in \mathcal{A} if and only if $\text{Ker}(N(C))$ is also a Lie subalgebra [5]. It can be directly checked that the latter is equivalent to (103).

Theorem 3.8. *$N(C)$ is a Nijenhuis tensor for the Lie algebra $(\mathcal{A}, \{\cdot, \cdot\})$ if and only if C satisfies the Yano-Ako conditions (103).*

Note that the above observation is closely related to the so called *coisotropic deformations* of associative structures as studied e.g. in [53] and that one can easily prove also quantum or supersymmetric analogs of the above theorem. More about Nijenhuis tensors for general brackets can be found in [5, 58]. Note finally that Nijenhuis tensors compatible with Poisson structures, *Poisson-Nijenhuis tensors*, provide a useful language for studying integrability of Hamiltonian systems [58, 71].

4 Double vector bundles and formalisms of Mechanics

The starting point of what follows is the observation that a skew (or Lie) algebroid can be described as a particular morphism of double vector bundles.

Definition 4.1. A *double vector bundle* is a manifold K with two compatible vector bundle structures, $\tau_i : K \rightarrow K_i$, $i = 1, 2$. The compatibility means that the Euler (Liouville) vector fields (generators of homotheties) associated with the two vector bundle structures commute.

This definition implies that, with every double vector bundle, we can associate the following diagram of vector bundles in which both pairs of parallel arrows form vector bundle morphisms:

$$\begin{array}{ccc} & K & \\ \tau_1 \swarrow & & \searrow \tau_2 \\ K_1 & & K_2 \\ \tau'_2 \searrow & & \swarrow \tau'_1 \\ & M & \end{array} \quad (106)$$

The above geometric definition (cf. [37, 38]) is a simplification of the original categorical concept of a double vector bundle due to Pradines [85], see also [67, 52].

Example 4.2. Let M be a smooth manifold and let (x^a) , $a = 1, \dots, m$, be a coordinate system in M . We denote by $\tau_M : TM \rightarrow M$ the tangent vector bundle and by $\pi_M : T^*M \rightarrow M$ the cotangent vector bundle. We have the induced (adapted) coordinate systems (x^a, \dot{x}^b) in TM and (x^a, p_b) in T^*M . Let $\tau : E \rightarrow M$ be a vector bundle and let $\pi : E^* \rightarrow M$ be the dual bundle. Let (e_1, \dots, e_n) be a basis of local sections of $\tau : E \rightarrow M$ and let (e^1, \dots, e^n) be the dual basis of local sections of $\pi : E^* \rightarrow M$. We have the induced coordinate systems:

$$\begin{aligned} (x^a, y^i), \quad y^i &= \iota(e^i), \quad \text{in } E, \\ (x^a, \xi_i), \quad \xi_i &= \iota(e_i), \quad \text{in } E^*. \end{aligned}$$

Thus we have the adapted local coordinates

$$\begin{aligned} (x^a, y^i, \dot{x}^b, \dot{y}^j) & \quad \text{in } \mathbb{T}E, \\ (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j) & \quad \text{in } \mathbb{T}E^*, \\ (x^a, y^i, p_b, \pi_j) & \quad \text{in } \mathbb{T}^*E, \\ (x^a, \xi_i, p_b, \varphi^j) & \quad \text{in } \mathbb{T}^*E^*. \end{aligned}$$

It is well known (cf. [51, 52, 94]) that the tangent bundle $\mathbb{T}E$ and the cotangent bundle \mathbb{T}^*E are canonical examples of double vector bundles:

$$\begin{array}{ccc} \mathbb{T}E & \xrightarrow{\tau_E} & E \\ \mathbb{T}\tau \downarrow & & \downarrow \tau \\ \mathbb{T}M & \xrightarrow{\tau_M} & M \end{array} \quad , \quad \begin{array}{ccc} \mathbb{T}^*E & \xrightarrow{\mathbb{T}^*\tau} & E^* \\ \pi_E \downarrow & & \downarrow \pi \\ E & \xrightarrow{\tau} & M \end{array} \quad (107)$$

with projections

$$\tau_E(x^a, y^i, \dot{x}^b, \dot{y}^j) = (x^a, y^i), \quad \mathbb{T}\tau(x^a, y^i, \dot{x}^b, \dot{y}^j) = (x^a, \dot{x}^b) \quad (108)$$

and

$$\pi_E(x^a, y^i, p_b, \pi_j) = (x^a, y^i), \quad \mathbb{T}^*\tau(x^a, y^i, p_b, \pi_j) = (x^a, \pi_j). \quad (109)$$

The corresponding pairs of commuting Euler vector fields are, respectively,

$$\nabla_1 = \dot{x}^a \partial_{\dot{x}^a} + \dot{y}^i \partial_{\dot{y}^i}, \quad \nabla_2 = y^i \partial_{y^i} + \dot{y}^j \partial_{\dot{y}^j}, \quad (110)$$

and

$$\nabla'_1 = p_a \partial_{p_a} + \pi_i \partial_{\pi_i}, \quad \nabla'_2 = p_a \partial_{p_a} + y^i \partial_{y^i}. \quad (111)$$

The fundamental fact we will explore is that the double vector bundles \mathbb{T}^*E^* and \mathbb{T}^*E are canonically isomorphic with an isomorphism

$$\mathcal{R}_\tau: \mathbb{T}^*E \longrightarrow \mathbb{T}^*E^* \quad (112)$$

being simultaneously an anti-symplectomorphism (we can choose a symplectomorphism as well) [51, 52, 42]. In local coordinates, \mathcal{R}_τ is given by

$$\mathcal{R}_\tau(x^a, y^i, p_b, \pi_j) = (x^a, \pi_i, -p_b, y^j). \quad (113)$$

This means that we can identify coordinates π_j with ξ_j , coordinates φ^j with y^j , and use the coordinates (x^a, y^i, p_b, ξ_j) in \mathbb{T}^*E and the coordinates (x^a, ξ_i, p_b, y^j) in \mathbb{T}^*E^* , in the full agreement with (112).

We known that skew algebroid structures on the vector bundle E correspond to linear bivector fields on E^* . As a matter of fact, a 2-contravariant tensor Π on E^* is *linear* if and only if the corresponding mapping $\tilde{\Pi}: \mathbb{T}^*E^* \rightarrow \mathbb{T}E^*$ is a morphism of double vector bundles. The commutative diagram

$$\begin{array}{ccc} \mathbb{T}^*E^* & \xrightarrow{\tilde{\Pi}} & \mathbb{T}E^* \\ \mathcal{R}_\tau \downarrow & \nearrow \varepsilon & \\ \mathbb{T}^*E & & \end{array} \quad (114)$$

describes a one-to-one correspondence between linear 2-contravariant tensors Π_ε on E^* and homomorphisms $\varepsilon : \mathbb{T}^*E \rightarrow \mathbb{T}E^*$ of double vector bundles covering the identity on E^* (cf. [52, 42]). In local coordinates, every ε as above is of the form

$$(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j) \circ \varepsilon = (x^a, \pi_i, \rho_k^b(x)y^k, c_{ij}^k(x)y^i\pi_k + \sigma_j^a(x)p_a) \quad (115)$$

which shows that it covers also $\rho : E \rightarrow \mathbb{T}M$ and corresponds to the linear tensor

$$\Pi_\varepsilon = c_{ij}^k(x)\xi_k\partial_{\xi_i} \otimes \partial_{\xi_j} + \rho_i^b(x)\partial_{\xi_i} \otimes \partial_{x^b} - \sigma_j^a(x)\partial_{x^a} \otimes \partial_{\xi_j} \quad (116)$$

on E^* . In [42], a (general) *algebroid* is defined as the above morphism ε of double vector bundles covering the identity on E^* , while a *skew algebroid* (resp., *Lie algebroid*) is such an algebroid for which the tensor Π_ε is skew-symmetric (resp., Poisson).

4.1 Lagrangian formalism for general algebroids

A generalized Lagrangian formalisms for Lie algebroids has been proposed by Liberman and Weinstein [61, 102] and developed in this setting by many authors (e.g. [9, 75, 76]). In [19, 21], in turn, has been observed that its geometric background is actually based on double vector bundle morphisms ε and the Jacobi identity plays no role in the construction of dynamics, that gives a space for further generalizations.

For a given an algebroid associated with the morphism $\varepsilon : \mathbb{T}^*E \rightarrow \mathbb{T}E^*$, a Lagrangian $L : E \rightarrow \mathbb{R}$ defines two smooth maps: the *Legendre mapping*: $\lambda_L : E \rightarrow E^*$, $\lambda_L = \tau_{E^*} \circ \varepsilon \circ dL$, and the *Tulczyjew differential* $\Lambda_L : E \rightarrow \mathbb{T}E^*$, $\Lambda_L = \varepsilon \circ dL$. On the diagram it looks like

$$\begin{array}{ccc} \mathbb{T}^*E & \xrightarrow{\varepsilon} & \mathbb{T}E^* \\ dL \uparrow & \nearrow \Lambda_L & \downarrow \tau_{E^*} \\ E & \xrightarrow{\lambda_L} & E^* \end{array} \quad (117)$$

The lagrangian function L defines the *phase dynamics* as the set $\mathcal{D} = \Lambda_L(E) \subset \mathbb{T}E^*$ which can be understood as an implicit differential equation on E^* , solutions of which are ‘phase trajectories’ of the system, $\beta : \mathbb{R} \rightarrow E^*$, and satisfy $t(\beta)(t) \in \mathcal{D}$, where t denotes the tangent prolongation of a C^1 -curve. An analog of the Euler-Lagrange equation for curves $\gamma : \mathbb{R} \rightarrow E$ is then

$$(E_L) : \quad t(\lambda_L \circ \gamma) = \Lambda_L \circ \gamma. \quad (118)$$

Equation (E_L) simply means that $\Lambda_L \circ \gamma$ is an admissible curve in $\mathbb{T}E^*$, thus it is the tangent prolongation of $\lambda_L \circ \gamma$. In local coordinates, \mathcal{D} has a parametrization by (x^a, y^k) via Λ_L in the form (cf. (115))

$$\Lambda_L(x^a, y^i) = \left(x^a, \frac{\partial L}{\partial y^i}(x, y), \rho_k^b(x)y^k, c_{ij}^k(x)y^i \frac{\partial L}{\partial y^k}(x, y) + \sigma_j^a(x) \frac{\partial L}{\partial x^a}(x, y) \right) \quad (119)$$

and equation (E_L) , for $\gamma(t) = (x^a(t), y^i(t))$, reads

$$(E_L) : \quad \frac{dx^a}{dt} = \rho_k^a(x)y^k, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial y^j} \right) = c_{ij}^k(x)y^i \frac{\partial L}{\partial y^k} + \sigma_j^a(x) \frac{\partial L}{\partial x^a}. \quad (120)$$

As one can easily see from (120), solutions are automatically admissible curves in E , i.e. $\rho^\Pi(\gamma(t)) = t(\tau \circ \gamma)(t)$. Since a curve in the canonical Lie algebroid $\mathbb{T}M$ is admissible if and only if it is a tangent prolongation of its projection to M , first-order differential equations

for admissible curves in $\mathbb{T}M$ may be viewed as certain second-order differential equations for curves in M . This explains why, classically, the Euler-Lagrange equations are usually viewed as second-order equations.

Remark. In the standard case, $E = \mathbb{T}M$, the Tulczyjew differential $\Lambda_L : \mathbb{T}M \rightarrow \mathbb{T}\mathbb{T}^*M$ is sometimes called the *time evolution operator* K (see [2]), as the first ideas of this operator go back to a work by Kamimura. This operator has been studied by several authors in many variational contexts, however, without recognition of its direct relation to a (Lie) algebroid structure. We named this map after Tulczyjew, since the above picture of the Lagrangian formalism is based on his ideas [91].

Example 4.3. There are many examples based on Lie algebroids, e.g. [9, 47, 75]. In particular, for the canonical Lie algebroid and the corresponding morphism which is the inverse of the Tulczyjew isomorphism [92]

$$\varepsilon = \alpha_M^{-1} : \mathbb{T}^*\mathbb{T}M \rightarrow \mathbb{T}\mathbb{T}^*M, \quad (121)$$

with $y^a = \dot{x}^a$, we get the traditional Euler-Lagrange equations

$$\frac{dx^a}{dt} = \dot{x}^a, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^a} \right) = \frac{\partial L}{\partial x^a}. \quad (122)$$

For a Lie algebroid which is just a Lie algebra with structure constants c_{ij}^k with respect to a chosen basis, we get the Euler-Poincaré equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial y^j} \right) = c_{ij}^k y^i \frac{\partial L}{\partial y^k}. \quad (123)$$

The above examples are associated with Lie algebroids, but the presence of some "non-holonomic constraints" may lead to Lagrangian systems on skew algebroids which are not Lie. This is related to 'quasi-Poisson brackets' associated with nonholonomic constraints [72, 95].

Example 4.4. (Skew algebroid of linear constraints) Consider an algebroid structure on a vector bundle E equipped with a Riemannian metric $\langle \cdot, \cdot \rangle_E$ and a vector subbundle C of E (linear constraints). Let $P : E \rightarrow C$ be the orthogonal projection. We can choose a local basis of orthonormal sections $(e_i) = (e_\alpha, e_A)$ of E such that (e_α) is a basis of local sections of C and, identifying E with E^* , consider the corresponding affine coordinates $(x^a, y^k) = (x^a, y^\alpha, y^A)$ on E . According to the *d'Alembert principle*, $\delta L(\mathbf{t}(\gamma)(t)) \in C^0$, where $C^0 \subset E^*$ is the annihilator of C , the constrained dynamics is locally written (cf. (120)) as

$$y^A = 0, \quad \frac{dx^a}{dt} = \rho_\alpha^a(x) y^\alpha, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial y^\beta} \right) - c_{\alpha\beta}^k(x) y^\alpha \frac{\partial L}{\partial y^k} - \sigma_\beta^a(x) \frac{\partial L}{\partial x^a} = 0. \quad (124)$$

If we deal with a Lagrangian of "mechanical type",

$$L = \frac{1}{2} (y^i)^2 - V(x), \quad (125)$$

then $\frac{\partial L}{\partial y^A} = y^A = 0$ and equations (124) reduce to

$$y^A = 0, \quad \frac{dx^a}{dt} = \rho_\alpha^a(x) y^\alpha, \quad \frac{dy^\beta}{dt} = c_{\alpha\beta}^\gamma(x) y^\alpha y^\gamma - \sigma_\beta^a(x) \frac{\partial V}{\partial x^a}, \quad (126)$$

that can be viewed as Euler-Lagrange equations for the algebroid associated with the orthogonal projection of the tensor Π_ε onto C^* , according to the orthogonal decomposition $E^* = C^0 \oplus C^*$ [19]. Of course, even for E being a Lie algebroid, if C is not a Lie subalgebroid, the the projected tensor is not a Poisson tensor and we deal with mechanics on a general algebroid, in fact, a skew algebroid, since the projected Poisson tensor remains skew-symmetric.

4.2 Hamiltonian formalism for general algebroids

Note that the linear tensor Π_ε on E^* gives rise also to a kind of Hamiltonian formalism. In [42] and [83] one refers to a 2-contravariant tensor as to a *Leibniz structure*, that however may cause some confusion with the *Leibniz algebra* in the sense of Loday. Anyhow, in the presence of Π_ε , by the *Hamiltonian vector field* associated with a function H on E^* we understand the contraction $\text{id}_H \Pi_\varepsilon$ as in (74). Thus the question of the Hamiltonian description of the dynamics $\mathcal{D} \subset TE^*$ is the question if \mathcal{D} is the image of a Hamiltonian vector field, i.e.

$$\mathcal{D} = \tilde{\Pi}_\varepsilon(dH(E^*)). \quad (127)$$

Every such a function H we call a *Hamiltonian associated with the Lagrangian L* . However, it should be stressed that, since ε and Π_ε can be degenerate, we have much more freedom in choosing generating objects (Lagrangians and Hamiltonians) than in the symplectic case. For instance, the Hamiltonian is defined not up to a constant but up to a Casimir function of the tensor Π_ε and for the choice of the Lagrangian we have a similar freedom. However, in the case of a hyperregular Lagrangian, we recover the standard correspondence between Lagrangians and Hamiltonians [21]. All this can be put into one diagram called the *Tulczyjew triple*:

$$\begin{array}{ccccc} T^*E^* & \xrightarrow{\tilde{\Pi}_\varepsilon} & TE^* & \xleftarrow{\varepsilon} & T^*E \\ \pi_{E^*} \swarrow & & \tau_{E^*} \swarrow & & \pi_E \searrow \\ E & \xrightarrow{\rho} & TM & \xleftarrow{\rho} & E \\ \tau \swarrow & & \tau_M \swarrow & & \tau \searrow \\ E^* & \xrightarrow{id} & E^* & \xleftarrow{id} & E^* \\ \pi \swarrow & & \pi \swarrow & & \pi \searrow \\ M & \xrightarrow{id} & M & \xleftarrow{id} & M \end{array} \quad (128)$$

The left-hand side is Hamiltonian, the right-hand side is Lagrangian, and the phase dynamics lives in the middle. Note finally that the above formalisms can still be generalized to include constraints (cf. [20]) and that a rigorous optimal control theory on Lie algebroids can be developed as well [7, 29].

5 Kirillov brackets and QD-algebroids

From the geometric point of view, of a particular interest are brackets on the spaces of sections of vector bundles given in differential terms. As examples we can consider the Lie bracket of vector fields (as sections of TM) and the Poisson (or Legendre) bracket on $C^\infty(M)$ (viewed as the space of sections of the trivial bundle $M \times \mathbb{R} \rightarrow M$) for a symplectic (resp., contact) manifold M .

In [Ki], Kirillov introduced *local Lie algebra* brackets on line bundles over a manifold M as Lie brackets on their sections given by local operators. These brackets we will call *Kirillov brackets*. According to the Peetre Theorem [84], local operators are locally differential operators, so we can as well deal locally with brackets defined by bi-differential operators.

The fundamental fact discovered in [Ki] is that these operators have to be of the first order and then, locally, they reduce to the conformally symplectic Poisson or Lagrange brackets on the leaves of a certain generalized foliation of M . For the trivial bundle, i.e. for the algebra $C^\infty(M)$ of functions on M , the local brackets reduce to Jacobi brackets. Hence, the line bundles equipped with a Kirillov bracket are sometimes called *Jacobi bundles*.

Theorem 5.1. *Any Kirillov bracket on sections of the trivial bundle $M \times \mathbb{R}$ (i.e. on $C^\infty(M)$) is a Jacobi bracket.*

Note that in the above we view $C^\infty(M)$ as a $C^\infty(M)$ -module, not as an algebra! A pure algebraic version of the above result is also valid [23, Theorem 4.2]. In the purely algebraic context, we replace the algebra $C^\infty(M)$ with an associative commutative algebra \mathcal{A} , and the space $\text{Sec}(E)$ of sections of a vector bundle $\tau : E \rightarrow M$ with an \mathcal{A} -module \mathcal{E} . We can define linear differential operators $D : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ between two \mathcal{A} -modules as follows. Let us observe first that, for $f \in \mathcal{A}$, we can construct a new operator $\delta(f)D : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ as the ‘commutator’ $[D, m_f]$:

$$(\delta(f)D)(x) = D(fx) - fD(x). \quad (129)$$

Definition 5.1. We say that a \mathfrak{k} -linear operator $D : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ between two \mathcal{A} -modules is a *differential operator of order $\leq n$* if

$$\delta(f_0) \cdots \delta(f_n)D = 0 \quad \text{for all } f_0, \dots, f_n \in \mathcal{A}. \quad (130)$$

The set of all such linear differential operators will be denoted $\text{Diff}_n(\mathcal{E}_1; \mathcal{E}_2)$, or simply $\text{Diff}_n(\mathcal{E})$ if $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$.

Note that if \mathfrak{k} is of characteristic 0, the condition (130) can be replaced by

$$\delta(f)^{n+1}D = 0 \quad \text{for all } f \in \mathcal{A} \quad (131)$$

and that the idea of defining differential operators in this pure algebraic context goes back to Grothendieck and Vinogradov [100]. It is easy to see that a zero-order differential operator D is just a module homomorphism, i.e. an \mathcal{A} -linear map.

Problem. Prove that, for an associative commutative algebra \mathcal{A} with unit $\mathbf{1}$, any first-order differential operator $D : \mathcal{A} \rightarrow \mathcal{A}$ is of the form

$$D(g) = X(g) + fg \quad (132)$$

for a certain $f \in \mathcal{A}$ and $X \in \text{Der}(\mathcal{A})$.

In $\text{Diff}_0(\mathcal{E})$ there is a special class, $\mathcal{A}_{\mathcal{E}}$, of zero-order differential operators which are just multiplications m_f by elements f of \mathcal{A} . Hence, in $\text{Diff}_1(\mathcal{E})$ there is a special class, $\text{QD}(\mathcal{E})$, of operators D such that $\delta(f)D \in \mathcal{A}_{\mathcal{E}}$ for all $f \in \mathcal{A}$. We call them *derivative endomorphisms*, *quasi derivations*, or *covariant differential operators*. In other words, a \mathfrak{k} -linear operator $D : \mathcal{E} \rightarrow \mathcal{E}$ is a derivative endomorphism if and only if, for all $f \in \mathcal{A}$, there is $\widehat{D}(f) \in \mathcal{A}$ such that

$$D(fX) = fD(X) + \widehat{D}(f)X \quad (133)$$

for all $X \in \mathcal{E}$. Of course, if $\mathcal{E} = \mathcal{A}$ is the trivial module, any quasi derivation is actually a first-order differential operator on the algebra \mathcal{A} .

Problem. Let $D, D_1, D_2 \in \text{QD}(\mathcal{E})$. Prove that the commutator $[D_1, D_2]$ is again in $\text{QD}(\mathcal{E})$, that the map

$$\widehat{D} : \mathcal{A} \rightarrow \mathcal{A} \quad f \mapsto \widehat{D}(f) \quad (134)$$

is a derivation, and that $D \mapsto \widehat{D}$ is a homomorphism of the Lie algebra $\text{QD}(\mathcal{E})$ with the commutator bracket into the Lie algebra $\text{Der}(\mathcal{A})$ of derivations of \mathcal{A} . We call this map the *universal anchor map*.

For multilinear operators we define analogously the corresponding commutators with respect to the i 'th variable,

$$\delta_i(f)D(x_1, \dots, x_p) = D(x_1, \dots, fx_i, \dots, x_p) - fD(x_1, \dots, x_p), \quad (135)$$

and call a multilinear operator D to be of order $\leq n$ if $\delta_i(f_0) \cdots \delta_i(f_n)D = 0$ for all $f_0, \dots, f_n \in \mathcal{A}$ and all i . This actually means that D is of order $\leq n$ with respect to each variable separately.

Definition 5.2. A differential Loday bracket on \mathcal{A} is a Loday bracket on \mathcal{A} given by a bi-differential operator.

Proposition 5.3 ([34]). *If \mathcal{A} has no nontrivial nilpotent elements, then every differential Loday bracket on \mathcal{A} is actually of the order ≤ 1 , thus a Jacobi bracket.*

5.1 QD-algebroids

A non-trivial differential requirement for a bracket $[\cdot, \cdot]$ on sections of a vector bundle $\tau : E \rightarrow M$ is that the bracket is a quasi-derivation with respect to each variable separately, i.e. $[X, \cdot]$ and $[\cdot, X]$ are quasi-derivations for each $X \in \mathcal{E} = \text{Sec}(E)$. Hence, $[X, fY] = f[X, Y] + \rho(X)(f)Y$ and $[fX, Y] = f[X, Y] - \sigma(Y)(f)X$ for all $X, Y \in \mathcal{E}$ and all $f \in \mathcal{A} = C^\infty(M)$, where $X \mapsto \rho(X) \in TM$ and $Y \mapsto \sigma(Y) \in TM$ is, respectively, the *left* and the *right anchor map*. Such brackets we will call *QD-algebroid brackets*. All skew algebroid brackets and all Poisson brackets are of this type. The difference is that the anchor map is of order 0 for skew algebroids and of order 1 for a Poisson structure ($f \mapsto \rho(f)$ is just passing to the Hamiltonian vector field). If the anchor maps are of the order 0 (\mathcal{A} -linear), we speak about an *algebroid* (cf. [42]). One can prove the following, somehow unexpected, fact.

Theorem 5.2 ([45, 25]). *Every QD-algebroid of rank > 1 is an algebroid. In other words, the anchor maps may be of the order 1 on line bundles only.*

If the bracket is a Lie bracket, we will speak about a *Lie QD-algebroid* (resp., *Lie algebroid*). We can also consider *Leibniz QD-algebroid* (resp., *Leibniz algebroid*) requiring additionally only the Jacobi identity (8) without the skew-symmetry assumption. We will not use the term *Loday algebroid* in this case to avoid a confusion with another concept of a Loday algebroid [31].

Actually, there is no big difference between Leibniz and Lie QD-algebroids, as the Jacobi identity forces the skew-symmetry. In particular, the left anchor must be equal to the right anchor. We can sum up these results as follows.

Theorem 5.3 ([25, 34]). *Let E be a vector bundle over M and let $[\cdot, \cdot]$ be a bilinear bracket operation on the $C^\infty(M)$ -module $\mathcal{E} = \text{Sec}(E)$ which satisfies the Jacobi identity (8) and which is a quasi-derivation with respect to both arguments.*

- (a) *If $\text{rank}(E) > 1$, then there is a vector bundle morphism $\rho : E \rightarrow TM$ over the identity map on M such that $\rho([X, Y]) = [\rho(X), \rho(Y)]$ and*

$$[fX, gY] = fg[X, Y] + f\rho(X)(g)Y - g\rho(Y)(f)X, \quad (136)$$

for all $X, Y \in \mathcal{E}$, $f, g \in C^\infty(M)$. Moreover, $[X, Y](p) = -[Y, X](p)$ if $\rho_p \neq 0$.

- (b) *If $\text{rank}(E) = 1$, then the bracket is skew-symmetric and defines a Kirillov bracket which, locally, is equivalent to a Jacobi bracket (52).*

Corollary 5.4. *Lie QD-algebroids on E are exactly Lie algebroids if $\text{rank}(E) > 1$, and local Lie algebras in the sense of Kirillov if $\text{rank}(E) = 1$.*

Corollary 5.5. *A Lie algebroid on a vector bundle E of rank > 1 is just a Lie bracket on sections of E which is a quasi-derivation with respect to one (hence both) argument.*

6 Graded manifolds

Definition 6.1. By a *graded manifold* we will understand a supermanifold \mathcal{M} with a \mathbb{N}^k -gradation in the structure sheaf that agrees with the parity. This means that, for any weight $w \in \mathbb{N}^k$, the homogeneous functions of weight w have parity coinciding with the parity of the total weight $w = w_1 + \dots + w_k$. For $k = 1$, one can also think that there is an atlas whose local coordinates have integer weights: odd coordinates have odd weights, and even coordinates even weights, that are preserved by changes of coordinates. An \mathbb{N}^k -*manifold of degree* $d \in \mathbb{N}$ is a \mathbb{N}^k -graded manifold whose local coordinates have total weights $\leq d$. A *symplectic manifold of degree* $r \in \mathbb{N}^k$ is an \mathbb{N}^k -graded manifold equipped with a homogeneous symplectic form of degree r .

Remark 6.2. There are various, also more general, concepts of a graded manifold, but the above will be sufficient for our purposes. We will assume in this note that the graded manifolds are *complete*, i.e. the even coordinates of non-zero weights take all real values. This is to avoid considering, for instance, open subsets in vector spaces instead of the whole vector spaces. Basic concepts and facts concerning \mathbb{Z} -graded manifolds can be found in [101]. Note that \mathbb{N} -manifolds (called also *N-manifolds*) have been first studied by Ševera [87] and Roytenberg [86].

Let us remark that the \mathbb{N}^k -grading can be conveniently encoded by means of the collection of *weight vector fields* which are jointly *diagonalizable*, i.e. there is an atlas of charts with local coordinates (x^a) in which

$$\Delta^s = \sum_a w_a^s x^a \partial_{x^a}, \quad s = 1, \dots, n, \quad (137)$$

where $w_a^s = w^s(x^a) \in \mathbb{N}$. An \mathbb{N}^k -manifold is complete if and only if each weight vector field is complete, i.e. induces an action of the multiplicative \mathbb{R} .

Example 6.3. If $\tau : E \rightarrow M$ is a vector bundle, then $E[d]$ is an \mathbb{N} -manifold of degree d , if we consider the basic functions being of weight 0 and functions linear in fibers being of degree d . That the coordinate changes preserve the weights is equivalent to preserving the vector bundle structure. Thus, every \mathbb{N} -manifold of degree 1 is of the form $E[1]$ with the algebra of smooth functions $C^\infty(E[1]) = \mathbb{A}(E^*)$. The corresponding weight vector field is the Euler vector field.

Remark 6.4. As the Grassmann algebra $\mathbb{A}(E^*)$ can be understood as the algebra of smooth functions on the graded manifold $E[1]$ (an *N-manifold of degree 1* in the terminology of Ševera and Roytenberg [86, 87]), following [97, 98] we can view the de Rham derivative d^Π as a vector field of degree 1 on $E[1]$. This vector field is *homological*, $(d^\Pi)^2 = 0$, if and only if we are actually dealing with a Lie algebroid. In local supercoordinates (x, \mathbf{y}) associated canonically with our standard affine coordinates (x, y) , we have

$$d^\Pi = \frac{1}{2} c_{ij}^k(x) \mathbf{y}^j \mathbf{y}^i \partial_{\mathbf{y}^k} + \rho_i^b(x) \mathbf{y}^i \partial_{x^b}. \quad (138)$$

Example 6.5. (Symplectic \mathbb{N} -manifolds of degree 1) A symplectic manifold of degree 1 is an \mathbb{N} -manifold of degree 1, thus $E[1]$ for a vector bundle E over M , equipped with a symplectic form of degree 1. It is easy to see that in this case E has to be linearly symplectomorphic to the cotangent bundle T^*M with the canonical symplectic form. The corresponding (super)Poisson bracket $\{\cdot, \cdot\}$ on $C^\infty(T^*[1]M)$ coincides with the Schouten bracket of multivector fields on M .

Functions of degree 2 (quadratic) correspond to bivector fields Λ on M . Moreover, the bracket is odd and the homological condition, $\{\Lambda, \Lambda\} = 0$, means that Λ is a Poisson tensor. The derived bracket, $\{f, g\}_\Lambda = \{\{f, \Lambda\}, g\}$, is closed on basic functions where it coincides (up to a sign) with the Poisson bracket of Λ .

6.1 The Big Bracket

Let $\tau : E \rightarrow M$ be a vector bundle and let $E[1]$ be the corresponding \mathbb{N} -manifold of degree 1. We will use local coordinates (x^a, y^i) in $E[1]$, where (x^a) are local coordinates (of weight 0) in a neighbourhood $W \subset M$ and y^i are linear functions in $\tau^{-1}(W) \subset E$ (of weight 1), corresponding to a basis of local sections of the dual bundle E^* . As we know, \mathbb{T}^*E is canonically a double vector bundle isomorphic to \mathbb{T}^*E^* , with the second bundle structure being $\mathbb{T}^*E \rightarrow E^*$. In consequence, \mathbb{T}^*E is canonically \mathbb{N}^2 -graded, with local coordinates (x^a, y^i, p_b, ξ_j) having bi-degrees $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, respectively. This \mathbb{N}^2 -grading comes from the canonical \mathbb{Z}^2 grading, in which (x^a, y^i, p_b, ξ_j) have bi-degrees $(0, 0)$, $(1, 0)$, $(0, 0)$, $(-1, 0)$, from the degree shift by $(1, 1)$ in the fibers of $\pi_E : \mathbb{T}^*E \rightarrow E$. The variables (x, p) are even and the variables (y, ξ) are odd. The canonical symplectic form has the bi-degree $(1, 1)$. The corresponding graded symplectic manifold \mathcal{M} we will denote $\mathbb{T}^*[(1, 1)]E[1]$. The double vector bundle structure yields canonical projections $\tau_0 : \mathbb{T}^*E \rightarrow E \oplus_M E^*$ and

$$\bar{\tau}_0 : \mathbb{T}^*[(1, 1)]E[1] \simeq \mathbb{T}^*[(1, 1)]E^*[1] \rightarrow (E \oplus_M E^*)[1]. \quad (139)$$

Of course, any double vector bundle is also an \mathbb{N} -graded manifold of degree 2. The corresponding weight vector field is the sum of the two commuting Euler vector fields. Therefore, the cotangent bundle $\mathbb{T}^*[2]E[1] \simeq \mathbb{T}^*[2]E^*[1]$ is canonically an \mathbb{N} -manifold of degree 2 with local coordinates (x^a, y^i, p_b, ξ_j) of degrees 0, 1, 2, 1, respectively. In particular, x^a, p_b are even coordinates and y^i, ξ_j are odd coordinates. The canonical symplectic form

$$\omega = dp_a dx^a + d\xi_i dy^i = -dx^a dp_a + dy^i d\xi_i \quad (140)$$

is homogeneous of degree 2. In both above cases, the corresponding graded Poisson bracket of degree $(-1, -1)$ (resp., -2), called sometimes the *big bracket*, is completely characterized locally by

$$\begin{aligned} \{p_b, x^a\} &= -\{x^a, p_b\} = \delta_b^a, & \{\xi_j, y^i\} &= \{y^i, \xi_j\} = \delta_j^i, \\ \{p_b, y^i\} &= \{p_b, \xi_j\} = 0, & \{x^a, y^i\} &= \{x^a, \xi_j\} = 0. \end{aligned} \quad (141)$$

Note that, for a vector space V , the big bracket on $\mathbb{T}^*[2]V[1] = (V \oplus V^*)[1]$ has been considered already by Kostant and Sternberg [59].

The projection (139) induces embeddings of the algebras of smooth functions $C^\infty(E[1]) = \mathbb{A}(E^*)$ and $C^\infty(E^*[1]) = \mathbb{A}(E)$ into $C^\infty(\mathbb{T}^*[(1, 1)]E[1])$ as functions of bi-degrees $(\bullet, 0)$ and $(0, \bullet)$ respectively. Moreover, functions of the total degree 1 on \mathcal{M} correspond to sections of $E \oplus_M E^*$.

Also the $\mathbb{A}(E^*)$ -module $\Phi_1(E) = \mathbb{A}(E^*) \otimes_{C^\infty(M)} \text{Sec}(E) = \text{Sec}(\wedge E^*; E)$ is therefore interpreted as spanned by functions of bi-degrees $(n, 1)$ on $\mathcal{M} = \mathbb{T}^*[(1, 1)]E[1]$, with $n \geq 0$. In local coordinates, elements of $\text{Sec}(\wedge^k E^*)$ are represented by polynomials

$$\sum_{i_1 < \dots < i_k} f_{i_1 \dots i_k}(x) y^{i_1} \dots y^{i_k}, \quad (142)$$

and elements of $\Phi_1^k(E)$ by polynomials

$$\sum_{j, i_1 < \dots < i_k} g_{i_1 \dots i_k}^j(x) y^{i_1} \dots y^{i_k} \xi_j. \quad (143)$$

As the canonical symplectic bracket is of the bi-degree $(-1, -1)$, it is closed on $\Phi_1(E)$ which is therefore a (graded) Lie subalgebra of $C^\infty(\mathcal{M})$.

Problem. Check that the big bracket restricted to $\Phi_1(E)$ is exactly the (generalized) Nijenhuis-Richardson bracket.

Remark 6.6. The big bracket is also closed on $\mathbb{A}(E) \otimes_{C^\infty(M)} \mathbb{A}(E^*) = \mathbb{A}(E \oplus_M E^*)$. It coincides (cf. [56]) with a bracket considered by Buttin [4]. She considered the commutator bracket of graded differential operators i_K on $\mathbb{A}(E^*)$ associated with elements K of $\mathbb{A}(E \oplus_M E^*)$ by

$$i_{\mu \otimes X}(\nu) = \mu \wedge i_X \nu. \quad (144)$$

6.2 The de Rham derivative as a homological vector field

Since, for $E = \mathbb{T}M$, the algebra of smooth functions $C^\infty(\mathbb{T}[1]M)$ is the algebra $\Omega(M)$ of differential forms, the de Rham derivative d , being a derivation in $\Omega(M)$, represents a vector field on $\mathbb{T}[1]M$. In local coordinates (x^a, \dot{x}^b) in $\mathbb{T}[1]M$ (here, x^a are even and \dot{x}^b are odd),

$$d = \dot{x}^a \partial_{x^a}. \quad (145)$$

This vector field is odd, so that $[d, d]_{\mathbb{T}[1]M} = 2d^2$, and homological, i.e. $[d, d]_{\mathbb{T}[1]M} = 0$. Its lift to the cotangent bundle $\mathbb{T}^*[(1, 1)]\mathbb{T}[1]M$, with local coordinates (x, \dot{x}, p, π) of bi-degrees $(0, 0), (1, 0), (1, 1)$, and $(0, 1)$ (and the total weights $(0, 1, 2, 1)$), respectively, reads

$$\mathcal{G}(d) = p_a \partial_{\pi_a} + \dot{x}^a \partial_{x^a}. \quad (146)$$

This is a Hamiltonian vector field with the cubic Hamiltonian of the bi-degree $(2, 1)$,

$$H_d = \dot{x}^a p_a. \quad (147)$$

More generally, if E is a Lie algebroid associated with a linear Poisson tensor

$$\Pi = \frac{1}{2} c_{ij}^k(x) \xi_k \partial_{\xi_i} \wedge \partial_{\xi_j} + \rho_i^b(x) \partial_{\xi_i} \wedge \partial_{x^b}, \quad (148)$$

then we can view the algebroid de Rham derivative d^Π as a vector field of degree 1 on $E[1]$. This vector field is *homological*, $(d^\Pi)^2 = 0$ [97, 98]. In local supercoordinates (x, y) associated canonically with our standard affine coordinates, we have

$$d^\Pi = \frac{1}{2} c_{ij}^k(x) y^j y^i \partial_{y^k} + \rho_i^b(x) y^i \partial_{x^b}. \quad (149)$$

The corresponding Hamiltonian of the lifted vector field reads

$$H = H_{d^\Pi} = \frac{1}{2} c_{ij}^k(x) y^j y^i \xi_k + \rho_i^b(x) y^i p_b. \quad (150)$$

The Hamiltonian is homological

$$\{H, H\} = 0, \quad (151)$$

and d^Π is of degree $(1, 0)$, so it defines the corresponding cohomology which can be restricted to any complex $\mathcal{A}^{(\bullet, n)}$. On $\mathcal{A}^{(\bullet, 0)} = \mathbb{A}(E^*)$, this cohomology is the classical Lie algebroid cohomology. On the other hand, for any section X of E , interpreted as an element in $\mathcal{A}^{(0, 1)}$, the function $\{H, X\}$ is of degree $(1, 1)$ and represents a linear vector field $d_1^\Pi X$ on E , the *complete lift* of X (cf. (75)) or, with a different interpretation, a linear vector field $\mathcal{G}^\Pi(X)$ on E^* , the *dual complete lift* of X .

6.3 The Frölicher-Nijenhuis bracket revisited

Theorem 3.3 implies immediately the following.

Proposition 6.7. *The bracket derived from the big bracket and the Hamiltonian H of d^Π ,*

$$\{K, L\}^H = \{\{K, H\}, L\}, \quad (152)$$

is closed on $\mathbb{A}(E)$ and coincides there with the generalized Schouten bracket $\llbracket \cdot, \cdot \rrbracket^\Pi$.

It is easy to see that the derived bracket (152) is closed also on vector-valued forms, i.e. on $\Phi_1(E^*)$. However, it gives not the Frölicher-Nijenhuis bracket, since it is not skew-symmetric. The Frölicher-Nijenhuis bracket differs from the derived one by a coboundary term (cf. (93)).

A tensor $N \in \Phi_1^1(E)$ we will call an (algebroid) *pseudo-Nijenhuis tensor* if $\{N, N\}^H = 0$.

Theorem 6.1 (cf. [6, 27]). *For $K \in \Phi_1^k(E^*)$ and $L \in \Phi_1^l(E^*)$, we have*

$$[K, L]^{FN} = \{K, L\}^H + (-1)^{k(l+1)} \{i_L K, H\}. \quad (153)$$

Any pseudo-Nijenhuis tensor N is weak-Nijenhuis, $\{H, N\}$ is a homological Hamiltonian, and the contracted bracket (101), corresponding to $\{H, N\}$, is again a Lie algebroid bracket.

For a discussion of brackets associated with the big bracket we refer to the survey article [56].

7 Courant bracket and Dirac structures

Recall that if (M, ω) is an $2n$ -dimensional symplectic manifold, then the symplectic form $\omega = \frac{1}{2}\omega_{ij}dx^i \wedge dx^j$ induces a vector bundle isomorphism

$$\tilde{\omega} : TM \ni V \mapsto -i_V \omega \in T^*M. \quad (154)$$

The inverse map

$$\tilde{\Lambda} = \tilde{\omega}^{-1} : T^*M \rightarrow TM \quad (155)$$

corresponds to a Poisson tensor $\Lambda = \frac{1}{2}\Lambda^{ij}\partial_{x^i} \wedge \partial_{x^j}$ via $\tilde{\Lambda}(\alpha) = i_\alpha \Lambda$. The fact that ω is closed, $d\omega = 0$, reads in coordinates as

$$(d\omega)_{kij} = \frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^j} = 0, \quad (156)$$

or, equivalently,

$$[\Lambda, \Lambda]^{kij} = \frac{\partial \Lambda^{ij}}{\partial x^k} \Lambda^{lk} + \frac{\partial \Lambda^{jk}}{\partial x^i} \Lambda^{li} + \frac{\partial \Lambda^{ki}}{\partial x^j} \Lambda^{lj} = 0. \quad (157)$$

Note that both equations, $d\omega = 0$ and $[\Lambda, \Lambda] = 0$, equivalent for invertible $\Lambda = \omega^{-1}$, make sense for an arbitrary 2-form ω and any bivector field Λ separately.

The (common) graph of ω and $\Lambda = \omega^{-1}$ is a vector subbundle L of the *Pontryagin bundle* $TM = TM \oplus_M T^*M$,

$$\begin{aligned} L_p &= \{(V_p + \zeta_p) \in T_p M \oplus T_p^* M : V_p = \tilde{\omega}(\zeta_p)\} \\ &= \{(V_p + \zeta_p) \in T_p M \oplus T_p^* M : \zeta_p = \tilde{\Lambda}(V_p)\}. \end{aligned} \quad (158)$$

The skew-symmetry of ω (or Λ) means that L is isotropic with respect to the canonical symmetric pseudo-Riemannian metric $\langle \cdot, \cdot \rangle_+$ on TM , where

$$\langle V_p + \zeta_p | U_p + \eta_p \rangle_\pm = \frac{1}{2} (\langle \zeta_p, U_p \rangle \pm \langle V_p, \eta_p \rangle). \quad (159)$$

The condition $d\omega = 0$ (or $[\Lambda, \Lambda] = 0$) means that L is involutive with respect to the *Courant bracket* [8] on $\mathcal{T}M$ defined by

$$[V + \zeta, U + \eta]_C = [V, U] + (\mathcal{L}_V \eta - \mathcal{L}_U \zeta + d\langle V + \zeta | U + \eta \rangle_-). \quad (160)$$

An important observation is that on any isotropic subbundle L the Courant bracket coincides with the *Dorfman bracket* [10] given by

$$[V + \zeta, U + \eta]_D = [V, U] + (\mathcal{L}_V \eta - i_U d\zeta). \quad (161)$$

Starting with a bivector field Λ and denoting, for arbitrary functions f, g , the corresponding Hamiltonian vector fields V_f, V_g , respectively, we get

$$[V_f + df, V_g + dg]_D = [V_f, V_g] + d\{f, g\}, \quad (162)$$

so that involutivity means $V_{\{f, g\}} = [V_f, V_g]$, that is equivalent to $[\Lambda, \Lambda] = 0$.

The Dorfman bracket is not skew-symmetric but it satisfies the Jacobi identity, so it is a Loday bracket. The Courant bracket is skew-symmetric but it does not satisfy the Jacobi identity; there is a *Jacobi anomaly*. Both brackets coincide on any isotropic subbundle L and give a Lie algebroid bracket on the space \mathcal{L} of its sections if L is involutive. Actually, the Dorfman bracket is a derived bracket. Namely, we use the Hamiltonian (147) to define a derived bracket out of the canonical Poisson bracket on $\mathbb{T}^*[2]\mathbb{T}[1]M$:

$$[[A, B]] = \{A, B\}^{H_d} = \{\{A, H_d\}, B\}. \quad (163)$$

This bracket is of degree -1 , so it is closed on functions of degree 1, thus sections of $\mathcal{T}M = \mathbb{T}M \oplus_M \mathbb{T}^*M$, where it coincides with the Dorfman bracket.

Definition 7.1. A *Dirac structure* is a maximal isotropic and involutive subbundle L of $(\mathcal{T}M, [\cdot, \cdot]_D)$. We call a vector field V on M an L -Hamiltonian vector field with an L -Hamiltonian function f if $(V + df) \in L$.

Let \mathfrak{H} be the set of all L -Hamiltonian vector fields, \mathfrak{H}_0 be the set of all L -Hamiltonian vector fields with the Hamiltonian 0, and \mathfrak{A} be the set of projections of sections of L onto $\mathbb{T}M$.

Theorem 7.1. The families \mathfrak{H} , \mathfrak{H}_0 , and \mathfrak{A} are Lie algebras of vector fields. Moreover, there is a canonical Poisson bracket on the space \mathfrak{H} of all Hamiltonians,

$$\{f_1, f_2\}_{\mathcal{L}} = \langle V_1, df_2 \rangle \quad \text{if } (V_i + df_i) \in \mathcal{L}, \quad (164)$$

that endows this space with a Lie algebra structure. If V_i is an L -Hamiltonian vector field with an L -Hamiltonian f_i , $i = 1, 2$, then $\{f_1, f_2\}_{\mathcal{L}}$ is an L -Hamiltonian of the L -Hamiltonian vector field $[V_1, V_2]$.

Dirac structures induce presymplectic foliations on M as follows.

Theorem 7.2. The Lie algebra of vector fields \mathfrak{A} induces a (generalized) foliation \mathcal{F} of M . Every leaf Y of this foliation is a presymplectic manifold with the closed two-form ω_Y induced from the map

$$\mathfrak{A} \times \mathfrak{A} \ni (V_1, V_2) \mapsto \Omega_L(V_1, V_2) = \zeta_1(V_2). \quad (165)$$

Here, ζ_1 is any 1-form satisfying $(V_1 + \zeta_1) \in \mathcal{L}$. Moreover, L -Hamiltonians are functions constant along the characteristic distributions of these presymplectic forms and the corresponding L -Hamiltonian vector fields are their Hamiltonian vector fields with respect to the presymplectic forms.

Example 7.2. (Dirac constrains) Dirac structures on manifolds provide a geometric setting for Dirac's theory of constrained mechanical systems. Let $Y \subset M$ be a submanifold determined by r independent constraints

$$\phi_1(p) = \cdots = \phi_r(p) = 0. \quad (166)$$

Note that the map $\phi = (\phi_i)$ defines actually a foliation $\mathcal{F} = \{\phi = \text{const}\}$, not a single submanifold. Let $\mathbb{T}\mathcal{F} \subset \mathbb{T}M$ be the corresponding distribution and $(\mathbb{T}\mathcal{F})^0 \subset \mathbb{T}^*M$ its annihilator (spanned by $d\phi_i$). The collective constraint ϕ defines a Dirac structure $L^\phi \subset \mathcal{T}M$ with the fibers

$$L_p^\phi = \{(V_p + \zeta_p) \in \mathcal{T}M : V_p \in \mathbb{T}_p\mathcal{F} \text{ and } \zeta \in \tilde{\omega}(V_p) + (\mathbb{T}\mathcal{F})_p^0\}. \quad (167)$$

In this case, L^ϕ -Hamiltonians are functions f satisfying

$$\{f, \phi_i\} = \mu^j \{\phi_j, \phi_i\}, \quad \mu^j \in C^\infty(M). \quad (168)$$

If ϕ consists of *first-class constraints*, $\{\phi_i, \phi_j\} = 0$, then L^ϕ -Hamiltonians are *first-class functions*, $\{f, \phi_i\} = 0$, and the Poisson bracket on the algebra of first-class functions is the original symplectic Poisson bracket. The bracket of first-class functions is again first-class:

$$\{\{f, g\}, \phi_i\} = \{\{f, \phi_i\}, g\} + \{f, \{g, \phi_i\}\} = 0. \quad (169)$$

If ϕ are *second-class constraints*, i.e. the matrix $(\{\phi_i, \phi_j\})$ is invertible, $(\{\phi_i, \phi_j\})^{-1} = (c^{ij})$, then it defines a foliation into symplectic submanifolds, so any function is L^ϕ -Hamiltonian and the Poisson bracket on the algebra of L^ϕ -Hamiltonians is the *Dirac bracket*

$$\{f, g\}_{\mathcal{L}^\phi} = \{f, g\} - \{f, \phi_i\} c^{ij} \{\phi_j, g\}. \quad (170)$$

7.1 Multi-Dirac and Poly-Dirac structures

The Dorfman bracket (161) can be immediately generalized (cf. [3]) to a bracket on sections of $\mathcal{T}^\bullet M = \mathbb{T}M \oplus_M \wedge^\bullet \mathbb{T}^*M$, where

$$\wedge^\bullet \mathbb{T}^*M = \bigoplus_{k=0}^{\infty} \wedge^k \mathbb{T}^*M, \quad (171)$$

so that sections of $\mathcal{T}^\bullet M$ are of the form $(X + \omega)$, where X is a vector field and ω is a differential form. The bracket, which we will call the *Grassmann-Dorfman bracket*, is formally given by the same formula (161) and it is also a Loday bracket. It can be reduced to a bracket $[[\cdot, \cdot]]^n$ on sections of the *Pontryagin Bundle of degree n* , i.e. the bundle $\mathcal{T}^n M = \mathbb{T}M \oplus_M \wedge^n \mathbb{T}^*M$, $n \in \mathbb{N}$, the *Grassmann-Dorfman bracket of degree n* , being an example of a *Loday algebroid bracket* [31]. In particular, the projection $\rho : \mathcal{T}M \rightarrow \mathbb{T}M$ onto the first summand yields the left anchor of the bracket.

Note that the Grassmann-Dorfman bracket is a part of the *graded Courant bracket* introduced in [99] on sections of $\wedge^\bullet \mathbb{T}M \oplus_M \wedge^\bullet \mathbb{T}^*M$. We will not discuss the latter generalization closer, as the Grassmann-Dorfman bracket will be sufficient for our purposes. On $\mathcal{T}^\bullet M$ we have another canonical structure, namely the non-degenerate symmetric pairing with values in $\wedge^\bullet \mathbb{T}^*M$,

$$\langle X + \omega, Y + \eta \rangle = \frac{1}{2} (i_X \eta + i_Y \omega), \quad (172)$$

where $\omega, \eta \in \Omega(M)$. This pairing is non-degenerate also on every $\mathcal{T}^n M$.

Definition 7.3. A vector subbundle L of the Pontryagin bundle of degree n is called a *multi-Dirac structure of degree n* if it is maximally isotropic with respect to the above pairing and *involutive*, i.e. whose sections are closed with respect to the Grassmann-Dorfman bracket.

The following is well known (see e.g. [3]).

Proposition 7.4. *The graph*

$$\mathfrak{G}(\alpha) = \{X + i_X \alpha : X \in \mathcal{T}M\} \subset \mathcal{T}^n M \quad (173)$$

of an $(n+1)$ -form α on M is a maximally isotropic subbundle in $\mathcal{T}^n M$. It is involutive if and only if α is closed. The form is non-degenerate if and only the projection of $\mathfrak{G}(\alpha)$ on the second summand is injective.

Note that closed non-degenerate $(n+1)$ -forms are sometimes called *n -plectic (multisymplectic) structures*. This justifies the following.

Definition 7.5. A *multi-Poisson structure of degree n* is a multi-Dirac structure of degree n which is the graph of a map $\wedge^n \mathcal{T}^*M \supset D \rightarrow \mathcal{T}M$ (on a vector subbundle domain D).

We can slightly generalize the above concepts by considering, for a real vector space W , the W -valued Grassmann-Dorfman bracket and the pairing as follows. The W -valued Grassmann-Dorfman bracket is defined on sections of $\mathcal{T}_W M = \mathcal{T}M \oplus_M (\wedge^\bullet \mathcal{T}^*M \otimes W)$ by

$$[[X + \omega \otimes a, Y + \eta \otimes b]]_W = [X, Y] + \mathcal{L}_X \eta \otimes b - i_Y d\omega \otimes a \quad (174)$$

and the W -valued pairing

$$\langle \cdot, \cdot \rangle_W : \mathcal{T}_W M \times_M \mathcal{T}_W M \rightarrow \mathcal{T}_W M \otimes W \quad (175)$$

by

$$\langle X + \omega \otimes a, Y + \eta \otimes b \rangle_W = \frac{1}{2} (i_X \eta \otimes b + i_Y \omega \otimes a) . \quad (176)$$

It is clear that *W -valued poly-Dirac structure of degree n* should be understood as maximal isotropic and involutive subbundles in $\mathcal{T}_W^n M$. If $W = \mathbb{R}^k$, we will speak about *poly-Dirac structures*. An example is given by the graph of a *W -valued polysymplectic form* $\alpha \in \Omega^2(M) \otimes W$ (called just *polysymplectic* if $W = \mathbb{R}^k$, cf. [14, 44]) which is a W -valued poly-Dirac structure (of degree 1). This justifies the following definition which agrees with the concept of a poly-Poisson structure studied in [48].

Definition 7.6. A *W -valued poly-Poisson structure of degree n* is a W -valued poly-Dirac structure of degree n which is the graph of a map $\wedge^n \mathcal{T}^*M \otimes W \supset D \rightarrow \mathcal{T}M$ (on a vector subbundle domain D).

Actually, we can replace $\mathcal{T}M$ with an arbitrary Lie algebroid E and replace $\mathcal{T}_W^n M$ with the Lie algebroid W -valued Pontryagin bundle of degree n ,

$$\mathcal{P}_W^n E = E \oplus_M (\wedge^n E^* \otimes W) . \quad (177)$$

7.2 Courant algebroids

Algebraic properties of the Courant bracket led to the concept of *Courant algebroid*. The original idea of Liu, Weinstein, and Xu [63] was based on the observation that $\mathcal{T}M$, endowed with the Courant bracket, plays the role of a ‘double’ object in the sense of Drinfeld [11] for a pair of Lie algebroids. Let us recall that, in complete analogy with Drinfeld’s Lie bialgebras, in the category of Lie algebroids there also exist ‘bi-objects’, Lie bialgebroids, introduced by Mackenzie and Xu [69]. On the other hand, every Lie bialgebra has a double which is a Lie algebra. This is not so for general Lie bialgebroids. Instead, Liu, Weinstein, and Xu showed that the double of a Lie bialgebroid is a more complicated structure they call a *Courant algebroid*, $\mathcal{T}M$ with the Courant bracket being a special case. In the general case:

- the Pontryagin bundle $\mathcal{T}M$ with the canonical symmetric pairing is replaced with a vector bundle $E \rightarrow M$ equipped with a nondegenerate symmetric bilinear form (\cdot, \cdot) on the bundle;
- the Courant bracket is replaced with a skew-symmetric bracket $[\cdot, \cdot]$ on $\text{Sec}(E)$;
- the canonical projection $\mathcal{T}M \rightarrow TM$ is replaced by a bundle map $\rho : E \rightarrow TM$. It induces a map $\mathcal{D} : C^\infty(M) \rightarrow \text{Sec}(E)$ defined by $\mathcal{D} = \frac{1}{2}\beta^{-1}\rho^*d$, where β is the isomorphism between E and E^* given by the bilinear form. In other words,

$$(\mathcal{D}f, e) = \frac{1}{2}\rho(e)f. \quad (178)$$

Definition 7.7 (cf. [63]). A *Courant algebroid* is a vector bundle $E \rightarrow M$ equipped with a nondegenerate symmetric bilinear form (\cdot, \cdot) on the bundle, a skew-symmetric bracket $[\cdot, \cdot]$ on $\text{Sec}(E)$, and a bundle map $\rho : E \rightarrow TM$ (the *anchor*) such that:

1. For any $e_1, e_2, e_3 \in \text{Sec}(E)$, $[[e_1, e_2], e_3] + (\text{cyclic}) = \mathcal{D}T(e_1, e_2, e_3)$, where $T(e_1, e_2, e_3)$ is the function on the base M defined by

$$T(e_1, e_2, e_3) = \frac{1}{3}([e_1, e_2], e_3) + (\text{cyclic}); \quad (179)$$

2. for any $e_1, e_2 \in \text{Sec}(E)$, $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]$;
3. for any $e_1, e_2 \in \text{Sec}(E)$ and $f \in C^\infty(M)$,

$$[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - (e_1, e_2)\mathcal{D}f; \quad (180)$$

4. $\rho \circ \mathcal{D} = 0$, i.e. for any $f, g \in C^\infty(M)$, $(\mathcal{D}f, \mathcal{D}g) = 0$;
5. for any $e, h_1, h_2 \in \text{Sec}(E)$,

$$\rho(e)(h_1, h_2) = ([e, h_1] + \mathcal{D}(e, h_1), h_2) + (h_1, [e, h_2] + \mathcal{D}(e, h_2)). \quad (181)$$

In what follows we will give equivalent ‘user friendly’ definitions.

7.3 Courant algebroid *via* the Dorfman bracket

For a Courant algebroid, instead of the skew-symmetric bracket with the anomaly in the Jacobi identity, we can consider a bracket which, like the Dorfman bracket, is not skew-symmetric, but satisfies the Jacobi identity, i.e. which is a Loday bracket. This new operation on sections of E is defined by

$$e_1 \circ e_2 = [e_1, e_2] + \frac{1}{2} \mathcal{D}(e_1, e_2), \quad (182)$$

so that the Courant bracket is the skew-symmetrization of " \circ ",

$$[e_1, e_2] = \frac{1}{2} (e_1 \circ e_2 - e_2 \circ e_1). \quad (183)$$

The Jacobi anomaly vanishes and we can state an equivalent simplified definition of Courant algebroid as follows (see [35, 93]).

Definition 7.8. A *Courant algebroid* is a vector bundle $E \rightarrow M$ equipped with a nondegenerate symmetric bilinear form (\cdot, \cdot) on E and a Leibniz product (bracket) \circ on $\text{Sec}(E)$, together with a vector bundle map (the anchor) $\rho : E \rightarrow \text{TM}$, which are compatible with (\cdot, \cdot) , that is,

$$\rho(X)(Y, Z) = (X, Y \circ Z + Z \circ Y) \quad (184)$$

and

$$\rho(X)(Y, Z) = (X \circ Y, Z) + (Y, X \circ Z). \quad (185)$$

The latter invariance of the pairing (\cdot, \cdot) with respect to the left multiplication implies the standard property of the anchor map ρ :

$$\begin{aligned} X \circ (fY) &= f(X \circ Y) + \rho(X)(f)Y. \\ \rho(X \circ Y) &= [\rho(X), \rho(Y)]. \end{aligned} \quad (186)$$

Besides its simplicity, this definition allows for considering as well the symmetric form (\cdot, \cdot) being degenerate. Note finally that one can define Nijenhuis tensors also for Courant algebroids [5, 27, 57] that leads to the concept of ‘generalized geometries’ in the spirit of the Hitchin’s *generalized complex geometry* [46] (see also [43]).

7.4 Symplectic N-manifolds of degree 2

The following characterizations of symplectic N-manifolds of degree 2 and Courant algebroids as certain Hamiltonian systems are due to Roytenberg [86].

Theorem 7.3. *There is a one-to-one correspondence between symplectic N-manifolds of degree two, (\mathcal{M}, ω) , and vector bundles $\tau : E \rightarrow M$ equipped with a pseudo-Riemannian structure (\cdot, \cdot) , i.e. a symmetric non-degenerate two-form in fibers. The symplectic manifold \mathcal{M}_E associated with $(E, (\cdot, \cdot))$ is the pullback of $\mathbb{T}^*[2]E[1]$ with respect to the embedding $E \hookrightarrow E \oplus_M E^*$ given by $X \mapsto X + (X, \cdot)$, i.e. it completes the commutative diagram*

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathbb{T}^*[2]E[1] \\ \downarrow & & \downarrow \\ E[1] & \longrightarrow & (E \oplus E^*)[1] \end{array}$$

Moreover, the symplectic form ω is the pullback of the canonical symplectic form on $\mathbb{T}^*[2]E[1]$.

Theorem 7.4. *There is a one-to-one correspondence between Courant algebroids and symplectic N -manifolds of degree 2, (\mathcal{M}_E, ω) , equipped with a cubic homological Hamiltonian H , $\{H, H\} = 0$. In this correspondence, we identify sections of E with functions of degree 1 on \mathcal{M}_E , basic functions (functions on M) with functions of degree 0 on \mathcal{M}_E , and the pseudo-riemannian metric with the Poisson bracket, $(X, Y) = \{X, Y\}$. The (Dorfman) algebroid bracket on sections of E is the derived bracket $X \circ Y = \{\{X, H\}, Y\}$.*

Consider local coordinates (x^a, ζ^i, p_b) in \mathcal{M}_E corresponding to coordinates (x^a) on M and a local basis $\{e_i\}$ of sections of E such that $(e_i, e_j) = g_{ij} = \text{const}$, $e_i = g_{ij}\zeta^j$ interpreted as a linear function on E . Then, the symplectic form ω reads

$$\omega = dp_a dx^a + \frac{1}{2} g_{ij} d\zeta^i d\zeta^j, \quad (187)$$

and any cubic Hamiltonian is of the form

$$H = \zeta^i \rho_i^a(x) p_a - \frac{1}{6} \phi_{ijk}(x) \zeta^i \zeta^j \zeta^k. \quad (188)$$

For the corresponding Courant algebroid, the Dorfman bracket and the anchor are uniquely determined by

$$([e_i, e_j], e_k) = \phi_{ijk}(x), \quad \rho(e_i) = \rho_i^a(x) \partial_{x^a}. \quad (189)$$

8 Nambu-Poisson brackets

There are two main ways of generalizing the notion of a Lie algebra. One way, already discussed, is to drop the skew-symmetry assumption and consider Loday brackets. Another concept is due to Filippov, who developed a theory of brackets with more than two arguments, i.e. *n-ary brackets*. In [13], he proposed a definition of such structures which we shall call *Filippov algebras*, with a version of the Jacobi identity for n -arguments which we will call *Filippov identity*:

$$\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \sum_{k=1}^n \{g_1, \dots, \{f_1, \dots, f_{n-1}, g_k\}, \dots, g_n\}. \quad (190)$$

A *Filippov bracket* is a skew-symmetric n -ary bracket satisfying (190). Note that in the binary case ($n = 2$), the Filippov identity coincides with the Jacobi identity. Independently, Nambu [79], looking for generalized formulations of Hamiltonian Mechanics, found n -ary analogs of Poisson brackets for which Takhtajan [90] rediscovered the Filippov identity (and called it Fundamental Identity). This leads to the concept of a *Nambu-Poisson bracket*, defined on a commutative associative algebra, which is a Filippov bracket satisfying additionally the *Leibniz rule*:

$$\{f_1 f'_1, f_2, \dots, f_n\} = f_1 \{f'_1, \dots, f_n\} + \{f_1, \dots, f_n\} f'_1. \quad (191)$$

Example 8.1. On \mathbb{R}^m , the n -ary bracket operation

$$\{f_1, \dots, f_n\} = \det \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j \leq n}, \quad (192)$$

where $n \leq m$, is a Nambu-Poisson bracket. Actually, each nonsingular Nambu-Poisson n -ary bracket, with $n > 2$, is locally of this form [17, 32, 74].

It is now clear that we can combine both generalizations and define *Filippov-Loday* algebras as those which are equipped with n -ary brackets, not skew-symmetric in general, but satisfying the Filippov identity. We can also define a Loday version of Nambu-Poisson algebras or rings.

Definition 8.2. Let \mathcal{A} be an associative commutative algebra. An n -ary bracket on \mathcal{A} is called a *Nambu-Loday* bracket if it satisfies the Filippov identity (190) and the Leibniz rule with respect to each argument $i = 1, \dots, n$:

$$\{f_1, \dots, f_i f'_i, \dots, f_n\} = f_i \{f_1, \dots, f'_i, \dots, f_n\} + \{f_1, \dots, f_i, \dots, f_n\} f'_i. \quad (193)$$

We encountered an unexpected phenomenon while looking for canonical examples of Nambu-Loday brackets. One can show that, for a wide variety of associative commutative algebras, including algebras of smooth functions, we get nothing more than what we already know, since Nambu-Loday brackets have to be skew-symmetric automatically. In particular, we can skip requiring the skew-symmetry in the standard definition of a Nambu-Poisson bracket. Recall that we have obtained a similar negative result for a Loday-type generalization of Lie algebroids (Theorem 5.3).

Theorem 8.1 ([33]). *If \mathcal{A} is an associative commutative algebra over a field of characteristic 0 and \mathcal{A} contains no nilpotents, then every Nambu-Loday bracket on \mathcal{A} is skew-symmetric. In particular, any Nambu-Loday bracket on $C^\infty(M)$ is a Nambu-Poisson bracket.*

For a deeper discussion of n -ary brackets we refer to the review paper [1].

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